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INFINITE ENERGY HARMONIC MAPS AND DEGENERATION OF HYPERBOLIC SURFACES IN MODULI SPACE

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1. Introduction

Let $(M, \sigma|dz|^2)$ be a closed, compact surface of genus g equipped with the (hyperbolic) metric σ of constant curvature $K \equiv -1$. If $\mathscr{M}_{-1}$ denotes the space of all constant curvature metrics on M, then both the group of orientation preserving diffeomorphisms Diff^+ and the group of diffeomorphisms isotopic to the identity Diff_0 act by pull-back on $\mathscr{M}_{-1}$; we define the Teichmüller space of genus g, T_g, to be the quotient space $\mathscr{M}_{-1}/\mathrm{Diff}_0$ and the moduli space of genus g, $\mathscr{M}_g$, to be the quotient space $\mathscr{M}_{-1}/\mathrm{Diff}^+$. Then $(M, \sigma|dz|^2)$ represents a point in the Teichmüller space T_g of surfaces of genus g. In [17], Sampson described a parametrization for a neighborhood of $(M, \sigma|dz|^2)$ in T_g, in terms of a neighborhood of zero in the vector space $QD(\sigma)$ of holomorphic quadratic differentials on $(M, \sigma|dz|^2)$. In [22], we used harmonic maps to derive an explicit asymptotic series for the hyperbolic metrics near $(M, \sigma|dz|^2)$ and, in Theorem 2.2, we shall show that this series converges, thus giving an explicit description of the real-analytic structure of the moduli space near an interior point $(M, \sigma|dz|^2)$.

The moduli space $\mathscr{M}_g$ admits a compactification $\overline{\mathscr{M}}_g$, which is a V-manifold without boundary; we use the notation $\mathscr{D}_g$ for the compactification divisor $\mathscr{D}_g = \overline{\mathscr{M}}_g \sim \mathscr{M}_g$. An element of $\mathscr{D}_g$ can be thought of as a Riemann surface with nodes, a connected complex space where points have neighborhoods complex isomorphic to either $\{|z| < \varepsilon\}$ (regular points) or $\{zw = 0; |z|, |w| < \varepsilon\}$ (nodes) and for which each component of the complement of the nodes has negative Euler characteristic.

Thus each component of the complement of the nodes admits a complete hyperbolic metric, with a deleted neighborhood of a node being isometric to two copies of a neighborhood of a hyperbolic cusp, given for

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example by the metric $ds^2 = |z|^{-2}(\log|z|)^{-2}|dz|^2$ on the punctured disk $\{0 < |z| < \varepsilon\}$. It is natural to think of noded surfaces arising as elements of $\mathscr{D}_g$ through a pinching process: consider a fixed family of simple closed curves $\gamma_1, \cdots, \gamma_n$ on M so that each component of the complement of the curves has negative Euler characteristic, and consider a family of hyperbolic metrics $\sigma_r|dz|^2$ on M. Suppose that the length $l_{\sigma_r}([\gamma_i])$ of the σ_r-geodesic representative of each of the free homotopy classes $[\gamma_i]$ of γ_i tends to zero as r tends to infinity. Then a subsequence of the points in $\mathscr{M}_g$ represented by the σ_r metrics would converge to a surface with n nodes. This noded surface would be topologically the result of identifying each curve γ_i to a point, the node. Bers [5] showed that this convergence was geometric in the sense that there are isometric copies of $\sigma_r|dz_r|^2$ on $M \sim \bigcup\{\gamma_i\}$ and a cusped hyperbolic metric $\sigma|dz|^2$ on $M \sim \bigcup\{\gamma_i\}$ representing a hyperbolic metric on a noded surface (in the complement of the nodes) so that $\sigma_r|dz_r|^2$ converges to $\sigma|dz|^2$ on $M \sim U\{\gamma_i\}$, uniformly on compacta.

Conversely, consider a surface $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ with hyperbolic metric $\sigma|dz|^2$ and nodes $p_1, \ldots, p_n$. A deformation of $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ will either keep the number and topological position of the nodes fixed and vary the remaining complex/hyperbolic structure, or it will "open" several of the nodes, replacing the neighborhood $\{zw = 0\}$ on M with the neighborhood $\{zw = t, \varepsilon > |t| > 0\}$ (see [16], [6], [24] for discussions), or it will do both. The deformed surface will represent a point in a deformation neighborhood in $\overline{\mathscr{M}}_g$ of the point in $\mathscr{D}_g$ represented by $(M, \sigma|dz|^2; p_1, \cdots, p_n)$; indeed, given a (small) deformation neighborhood $\mathscr{N}$ of the point in $\mathscr{D}_g$ represented by $(M, \sigma|dz|^2; p_1, \cdots, p_n)$, we can represent each point $[\rho] \in \mathscr{N}$ with a surface obtained by either fixing the nodes of $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ and varying the remaining complex/hyperbolic structure and/or opening some of the nodes $p_1, \cdots, p_n$.

In this paper, we investigate such a neighborhood $\mathscr{N}$ of a hyperbolic surface with nodes (or with paired cusps); because $\overline{\mathscr{M}}_g$ is a V-manifold and our deformations will be smooth, we work on a smooth deformation space $\widetilde{\mathscr{P}}_g$ which is ramified cover of a subset of $\overline{\mathscr{M}}_g$. Our basic method is to use harmonic maps from such a noded surface to nearby hyperbolic structures, and then to compare the resulting pull-back metrics using the analytic formulae resulting from the harmonicity of the map. In [16], Masur used quasiconformal maps and a cut-and-paste construction along with techniques of coherent sheaves to give a parametrization of such a deformation neighborhood in terms of meromorphic quadratic differentials on (M, σ) with at most second order poles at the nodes; we will rederive his result using the holomorphic differentials associated to harmonic maps. We then derive a converging asymptotic series for the hyperbolic metrics on the surfaces within the deformation neighborhood; this improves the previously mentioned result of Bers [5] that the hyperbolic metrics vary continuously in the deformation neighborhood.

This article is organized as follows. In §2, we fix our notation and derive our deformation theory for hyperbolic metrics varying in an "internal" neighborhood of the moduli space $\mathcal{M}_g$. In §3 we derive the necessary theory of harmonic maps from a noded surface to a surface without nodes. Specifically we prove

Theorem 3.11. *If $(M, \sigma|dz|^2, p)$ is a noded Riemann surface with metric $\sigma|dz|^2$ and node p, $(N, \rho|dw|^2)$ is a hyperbolic surface, and γ a simple closed curve on N, then in each homotopy class of maps $w: M \sim p \to N \sim \gamma$ which admits a diffeomorphism, there exists a diffeomorphism satisfying the Euler-Lagrange equation for a harmonic map, and mapping $M \sim p$ onto $N \sim \gamma_N$, where γ_N is the geodesic representative for the free homotopy class γ in the metric $\rho|dw|^2$. The quadratic differential $(w^*\rho)^{2,0}$ is holomorphic on $M \sim p$; at the node p, the differential $(w^*\rho)^{2,0}$ has a second order pole whose leading coefficient equals one-fourth the square ρ-length of γN. If $u: M \sim p \to N \sim \gamma_N$ is a diffeomorphism satisfying the Euler-Lagrange equation and having a holomorphic quadratic differential with a second order pole with real leading coefficient at the node, then $u = w$.*

Also in §3, we derive the analogous result for maps between noded surfaces which do not open a node; here we use results of Schoen-Yau [20] and Lohkamp [15]. In §4, we define the relevant smooth deformation space $\widehat{\mathcal{P}_g}$ and give a parametrization of a neighborhood of $\widehat{\mathcal{P}_g}$ of a noded surface $(M, \sigma|dz|^2, p_1, \cdots, p_n)$ in terms of meromorphic quadratic differentials on (M, σ) and twist parameters coming from Fenchel-Nielsen theory. Finally in §5, we show

Theorem 5.3. *Let $\widehat{\mathcal{N}}$ be a neighborhood in $\widehat{\mathcal{P}_g}$, parametrized by a choice of Fenchel-Nielsen coordinates $(\vec{l}, \vec{\theta})$. Let M be a smooth surface, possibly with the exception of some nodes. Then if $ds^2(\vec{l}, \vec{\theta})$ is a family of hyperbolic metrics on M representing a neighborhood of $(\vec{l}_0, \vec{\theta}_0)$ in $\widehat{\mathcal{N}}$, then $ds^2(\vec{l}, \vec{\theta})$ is real analytic in $\vec{l}, \vec{\theta}$.*

This is straightforward for $\widehat{\mathscr{N}} \subset \mathscr{M}_g$. Bers [5] showed that $ds^2(\vec{l}, \vec{\theta})$ is not only real analytic for $l_j > 0$ (for all j), but continuous when $l_j \geq 0$.

We conclude §5 by giving a somewhat explicit description of the Taylor series for $ds^2(\vec{l}, \vec{\theta})$ in terms of meromorphic quadratic differentials on the surface $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ (with nodes $p_1, \cdots, p_n$) and the operator $(\Delta_\sigma - 2)^{-1}$: the formal power series needs to be interpreted in light of there being a nontrivial kernel for $\Delta_\sigma - 2$ on a cusped surface $(M, \sigma|dz|^2)$.

2. Harmonic maps between surfaces

Let $(M, \sigma|dz|^2)$ and $(N, \rho|dw|^2)$ be smooth Riemannian surfaces; we write our metrics $\sigma|dz|^2$ and $\rho|dw|^2$ in isothermal coordinates. For a Lipschitz map $w: (M, \sigma|dz|^2) \to (N, \rho|dw|^2)$, we define the holomorphic energy at a point to be

$$\mathscr{H}(w, \sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2$$

and the antiholomorphic energy to be

$$\mathscr{L}(w, \sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)} w_{\bar{z}}|^2.$$

Then we define the energy density $e(w; \sigma, \rho)$ by $e(w; \sigma, \rho) = \mathscr{H} + \mathscr{L}$, the Jacobian of the map $\mathscr{J}(w; \sigma, \rho)$ by $\mathscr{J}(w; \sigma, \rho) = \mathscr{H} - \mathscr{L}$, and the total energy $E(w; \sigma, \rho)$ by

$$\begin{aligned} E(w; \sigma, \rho) &= \int_M e(w; \sigma, \rho)\sigma \, dz \, d\bar{z} \\ &= \int_M (|w_z|^2 + |w_{\bar{z}}|^2)\rho(w(z)) \, dz \, dz\bar{z}. \end{aligned}$$

From the last expression we see that the total energy depends upon the metric of the target surface, but only upon the conformal structure of the source. The functions $\mathscr{H}$, $\mathscr{L}$, e, and $\mathscr{J}$ depend upon the background source metric; while we will use several such source metrics, we will always employ the same symbol $\mathscr{H}$ to mean the holomorphic energy with respect to the metric in use, relying on the context to avoid confusion. We will also consistently use σ or σ_0 for a source metric, and ρ or ρ_t for a target metric.

The Euler-Lagrange equation for the energy functional is

$$\tau(w) = w_{z\bar{z}} + (\log \rho)_w w_z w_{\bar{z}} = 0;$$

the tensor $\tau(w)$ is called the tension, and a map which satisfies the equation $\tau(w) = 0$ is called harmonic. For our purposes a different point of view is useful: if we pull back the target metric form $\rho|dw|^2$ by the maps w, it decomposes as

$$\begin{aligned} w^*\rho|dw|^2 &= \rho w_z \overline{w_z}\, dz^2 + (\rho|w_z|^2 + \rho|w_{\bar z}|^2)\, dz\, d\bar z \\ &\quad + \overline{\rho w_z \overline{w_{\bar z}}}\, d\bar z^2 \\ &= \Phi\, dz^2 + \sigma e(w;\sigma,\rho)\, dz\, d\bar z + \overline{\Phi}\, d\bar z^2 \\ &= \Phi\, dz^2 + \sigma\left[\mathscr{H} + \frac{|\Phi|^2}{\sigma^2\mathscr{H}}\right] dz\, d\bar z + \overline{\Phi}\, d\bar z^2. \end{aligned} \tag{2.1}$$

If $\mathscr{J}(w;\sigma,\rho)$ does not vanish anywhere on M, then it is easy to show (see [17]) that w is harmonic $\Leftrightarrow$ $\Phi\, dz^2$ is a holomorphic quadratic differential on $(M, \sigma|dz|^2)$.

Define, on $(M, \sigma|dz|^2)$, the Laplace–Beltrami operator

$$\Delta_\sigma = \frac{4}{\sigma}\frac{\partial^2}{\partial z \partial \bar z}$$

and the Gauss curvature,

$$K(\sigma) = -\frac{2}{\sigma}\frac{\partial^2 \log \sigma}{\partial z \partial \bar z}.$$

Then one derives from the Euler-Lagrange equation the Bochner-type equation (see [19])

$$\Delta_\sigma \log \mathscr{H} = -2K(\rho)\mathscr{J} + 2K(\sigma) \tag{2.2a}$$

$$= -2K(\rho)\left[\mathscr{H} - \frac{|\Phi|^2}{\sigma^2\mathscr{H}}\right] + 2K(\sigma). \tag{2.2b}$$

If $K(\rho) \equiv -1$, as it will always be for us, we have the fundamental equation of study,

$$\Delta_\sigma \log \mathscr{H} = 2\mathscr{H} - \frac{2|\Phi|^2}{\sigma^2\mathscr{H}} + 2K(\sigma). \tag{2.2}$$

We now review the discussion of [22], which is based on the following construction. Let M be a closed, boundaryless surface of genus $g > 1$ and fix a hyperbolic metric $\sigma|dz|^2$ on it. Then, choose a point in the Teichmüller space T_g, and represent it by a hyperbolic metric $\rho|dw|^2$. There exists [7] a unique [11] harmonic diffeomorphism [17], [19] $w(\sigma,\rho)$: $(M, \sigma|dz|^2) \to (M, \rho|dw|^2)$ in the homotopy class of the identity map id: $M \to M$. By setting $\Phi dz^2 = (w(\sigma, \rho)^*(\rho|dw|^2))^{2,0}$, we find that we have a map

$$\Phi\colon T_g \to QD(\sigma),$$

where $QD(\sigma)$ denotes the space of holomorphic quadratic differentials on $(M, \sigma|dz|^2)$.

Theorem 2.1 [17], [22]. *Φ is a homeomorphism onto all of $QD(\sigma)$.*

Thus $QD(\sigma)$ provides global coordinates for T_g via the inverse map Φ^{-1}, and (2.2) is uniquely solvable for all parameter values $\Phi \in QD(\sigma)$.

If $w\colon (M, \sigma|dz|^2) \to (M, \rho|dw|^2)$ is harmonic, then so is the identity map id: $(M, \sigma|dz|^2) \to (M, w^*(\rho|dw|^2))$. Since T_g consists of classes of hyperbolic metrics equivalent under the action of the component of the identity of the diffeomorphism group, we might as well have chosen $w^*(\rho|dw|^2)$ to represent $[\rho|dw|^2] \in T_g$. So we can assume that id: $(M, \sigma|dz|^2) \to (M, \rho|dw|^2)$ is harmonic.

Consider the hyperbolic metrics $\rho_t|dw_t|^2$ determined by the ray $t\Phi_0$ in the coordinate space $QD(\sigma)$, where Φ_0 is an element of $QD(\sigma) \sim \{0\}$, and t is a real nonnegative number. We rewrite (2.2) as

$$\Delta_\sigma \log \mathscr{H}(t) = 2\mathscr{H}(t) - \frac{2t^2|\Phi_0|^2}{\sigma^2 \mathscr{H}(t)} - 2, \tag{2.3}$$

where we denote $\mathscr{H}(\mathrm{id}; \sigma, \rho_t)$ as $\mathscr{H}(t)$.

We want to write $\mathscr{H}(t)$ as a series in t depending on $|\Phi_0|^2/\sigma^2$ and the source surface $(M, \sigma|dz|^2)$. It is easy to see that $\mathscr{H}(0) \equiv 1$, and if we know $d/dt|_{t=0}\mathscr{H}(t), \cdots, d^n/dt^n|_{t=0}\mathscr{H}(t)$, after applying $d^{n+1}/dt^{n+1}|_{t=0}$ to (2.3) we can solve for $d^{n+1}/dt^{n+1}|_{t=0} \times \mathscr{H}(t)$. The maximum principle will force all the odd order derivatives to vanish [22], and we will be left with a series

(2.4)

$$\begin{aligned}\mathscr{H}(t) = 1 &+ \left[-2(\Delta_\sigma - 2)^{-1}\frac{2|\Phi_0|^2}{\sigma^2}\right]\frac{t^2}{2} \\ &+ (\Delta(_\sigma - 2)^{-1} + \left[2\left(-2(\Delta_\sigma - 2)^{-1}\frac{2|\Phi_0|^2}{\sigma^2}\right)^2\right. \\ &\left. + \frac{4|\Phi_0|^2}{\sigma^2}\left(-2(\Delta_\sigma - 2)^{-1}\frac{2|\Phi_0|^2}{\sigma^2}\right)\right]\frac{t^4}{24} + O(t^6),\end{aligned}$$

where all of the terms are computable. The only nonmultiplicative operator which arises is $(\Delta_\sigma - 2)^{-1}$ (which is well defined because Δ_σ is negative); it is a global self-adjoint operator which acts as multiplication by $-\frac{1}{2}$ on the constant functions.

Using the explicit expansion $\sum \mathscr{H}^{(n)}(t)t^n/n!$ and (2.1) we expand the hyperbolic metric $\rho_t|dw_t|^2$ as

$$\rho_t|dw|^2 = t\Phi_0\,dz^2 + \sigma\left\{1 + \left[\frac{2|\Phi_0|^2}{\sigma^2} + \left(-2(\Delta_\sigma - 2)^{-1}\frac{2|\Phi_0|^2}{\sigma^2}\right)\right]\frac{t^2}{2} + O(t^4)\right\} dz\,d\bar{z} + t\overline{\Phi}_0\,d\bar{z}^2,$$

where $O(t^4)$ represents the higher order terms of an explicit series.

We then ask whether this series converges.

Theorem 2.2. *$\mathscr{H}(t)$ is real-analytic in t, so that both series* (2.4) *and* (2.5) *converge for $|t|$ small.*

Proof. On a neighborhood $U \subset C^{2,\alpha}(M)\times\mathbf{C}$ of $(\mathscr{H}\equiv 1, 0)$ consider the mapping $F(\mathscr{H}, \tau)\colon C^{2,\alpha}\times\mathbf{C}\to C^\alpha(M)$ given by

$$F(\mathscr{H}, \tau) = \Delta_\sigma \log\mathscr{H} - 2\mathscr{H} + 2\tau^2\frac{|\Phi_0|^2}{\sigma^2\mathscr{H}} + 2.$$

Setting $\mathscr{H}(0)\equiv 1$, we see that

$$F(\mathscr{H}(0), 0) = 0$$

and we can compute that the linearization of F in a neighborhood of $(\mathscr{H}\equiv 1, \tau = 0)$ is

$$dF_{\mathscr{H}}(\mathscr{H}(0, 0)[\psi] = \Delta_\sigma\psi - 2\psi.$$

Thus $dF_{\mathscr{H}}$ is clearly invertible, so by the Analytic Implicit Function Theorem (cf. [4, Theorem 3.3.2, p. 134]), there exists one and only one solution $\widetilde{\mathscr{H}}(\tau)$ of $F(\mathscr{H}(\tau), \tau) = 0$ near $(\mathscr{H}(0), 0)$ that is a complex analytic function of τ for $|\tau|$ small.

However, we know that the solution $\mathscr{H}(t)$ of (2.3) constructed by Theorem 2.1 is unique; consequently that $\mathscr{H}(t)$ must agree with the solution $\mathscr{H}(t)$ constructed above, when $\tau = t$ is purely real. Thus, the solutions $\mathscr{H}(t)$ constructed by Theorem 2.1 must be real analytic, since they are the restriction of a complex analytic function to the real axis of the parameter domain.

Remarks. (i) The proof depends heavily on the compactness of M, especially for the invertibility of the operator $\Delta_\sigma - 2$.

(ii) Choose a complex basis $\Phi_1, \cdots, \Phi_{3g-3}$ for $QD(\sigma)$; then if $\vec{t} = (t_1, \cdots, t_{3g-3}) \in \mathbf{C}^{3g-3}$, $\Phi(\vec{t}) = \sum t_j\Phi_j$ defines local coordinates for

T_g. Thus a similar expansion and real analyticity result hold for $\mathscr{H}(\vec{t})$ defined in terms of $\Phi(\vec{t})$.

If $(M, |dz|^2)$ represents a hyperbolic metric on a compact surface, then we find an explicit real analytic expansion of the hyperbolic metric on the compact surfaces in a neighborhood of $(M, \sigma|dz|^2)$ in the Teichmüller space T_g, parametered by a neighborhood in the vector space of holomorphic quadratic differentials $QD(\sigma)$ on $(M, \sigma|dz|^2)$. This is the situation we want to generalize to the case where $(M, \sigma|dz|^2)$ is a surface with nodes, representing a point of $\mathscr{D}_g$.

3. Harmonic from noded surfaces

Recall that a Riemann surface $(M, z; p_1, \cdots, p_n)$ with nodes $p_1, \cdots, p_n$ is a connected complex space where $M \sim \bigcup\{p_i\}$ is a (possibly disconnected) Riemann surface, all of whose components admit a hyperbolic metric and for which a neighborhood of a node is complex isomorphic to $\{(\zeta, w) \in C^2 | \ \zeta w = 0, |\zeta| < \varepsilon, |w| < \varepsilon\}$; here z represents a generic conformal coordinate. Such a surface lies in the compactification divisor $\mathscr{D}_g = \overline{\mathscr{M}}_g \sim \mathscr{M}_g$.

We are interested in describing the harmonic maps from a Riemann surface with nodes $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ to a Riemann surface (possibly) with nodes $(N, \rho|dw|^2; q_1, \cdots, q_m)$, where $m \le n$, and $M \sim \bigcup\{p_1\}$ is homeomorphic to the complement, as in $N \sim \bigcup\{q_i\}$, of some disjoint simple closed curves.

For simplicity of exposition, we suppose that $(M, \sigma|dz|^2, p)$ has a single node p, i.e., a single pair of hyperbolic cusps; we also begin by assuming that $(N, \rho|dw|^2)$ has no nodes but has a simple closed geodesic γ so that $M \sim p$ is homeomorphic to $N \sim \gamma$. We will emphasize the case when the ρ-length of γ is small.

For now, we will consider $(M, \sigma|dz|^2; p)$ as a pinched limit of closed smooth surfaces $(M, \sigma_r|dz_r|^2)$ as described in §1; that we can do this follows from work of Earle-Marden [6]. We will show that the harmonic maps $w_r\colon (M, \sigma_r|dz_r|^2) \to (N, \rho|dw|^2)$ converge to a harmonic map $w\colon (M, \sigma|dz|^2; p) \to (N, \rho|dw|^2)$ as $r \to \infty$. Later we will consider the effect of having chosen $(M, \sigma|dz|^2; p)$ as the limit of a different family of surfaces.

Even for this problem we must be careful. In particular, it is not true that if we consider $(M, \sigma_r|dz_r|^2)$ as a family of *arbitrarily parametrized* surfaces leaving every compact set in T_g, then the harmonic maps w_r: $(M, \sigma_r|dz_r|^2) \to (N, s_\rho^2)$ will converge to a nice map $w: M \to N$. To see this, one need only consider the following boundary value problem, recalling that only the conformal structure of the source metric is important for the harmonic map problem. Let M be the cylinder $[-1, 1] \times [0, 1]$, where we identify the boundaries $[-1, 1] \times \{0\}$ and $[-1, 1] \times \{1\}$ and use the natural (x, y) coordinates, and let N be the cylinder $[-\cosh^{-1} 1/\varepsilon, \cosh^{-1} 1/\varepsilon] \times [0, 1]$, with boundary identifications

$$[-\cosh^{-1}\varepsilon^{-1}, \cosh^{-1}\varepsilon^{-1}] \times \{0\} \cong [-\cosh^{-1}\varepsilon^{-1}, \cosh^{-1}\varepsilon^{-1}] \times \{1\},$$

and natural (u, v) coordinates.

Then let $z_r = x_r + iy_r = rx + iy/r$, $\sigma_r = 1$, and $ds_\rho^2 = du^2 + (\varepsilon^2 \cosh^2 u)\, dv^2$. The harmonic map w_r: $(M, \sigma_r|dz_r|^2) \to (N, ds_\rho^2)$ which takes the boundaries $\{\pm 1\} \times [0, 1]$ to $\{\pm\cosh^{-1}\varepsilon^{-1}\} \times [0, 1]$ with boundary conditions $v_r(\pm 1, y) = y$ is given by $v_r(\pm 1, y) = y$ and $u_r(x, y) = u_r(x)$, where $u_r(x)$ satisfies the Euler-Lagrange equation $u^4 = r^4\varepsilon^2 \sinh 2u$. It is not hard to see that if $x \neq \pm 1$, $u_r(x) \to 0$ as $r \to \infty$. Thus, the limiting map collapses the interior of the domain onto the core geodesic $\{0\} \times [0, 1]$ of the range. Later we will consider a better parametrization of this problem.

Let $(M, \sigma|dz|^2; p)$ be a hyperbolic surface with a node p, and let $(N, \rho|dw|^2)$ be a closed surface of genus g without nodes equipped with a hyperbolic metric $\rho|dw|^2$. We now construct a harmonic map $w: (M, \sigma|dz|^2), p) \to (N, \rho|dw|^2)$ as the limit of harmonic maps w_r: $(M_r, \sigma_r|dz^\gamma|^2) \to (N, \rho|dw|^2)$. To do this, notice that a neighborhood U of p in $M \sim p$ is isometrically a pair of standard cusps in the Poincaré upper half-plane, i.e., U is two copies of $P_a = \{0 \le x \le 1, y > a | \{0\} \times y \cong \{1\} \times y\}$ with metric $\sigma|dz|^2 = |dz|^2/y^2$; we denote the images of P_a in $M \sim p$ also as P_a. Let $\lambda: (a, \infty) \to \mathbf{R}$ be a smooth function with equals 1 in a neighborhood of a and equals y^2 on $(a+1, \infty)$. Equip P_a with the metric $\sigma_0|dz|^2 = \lambda\sigma|dz|^2$; this new metric is conformally equivalent to the original Poincaré metric, but is flat on P_{a+1}. The metric $\sigma_0|dz|^2$ on P_a extends to a metric $\sigma_0|dz|^2$ on $M \sim p$. We then construct M_r by the following process. We remove the two copies of P_r from $(M, \sigma|dz|^2; p)$, obtaining a (possibly disconnected) manifold with boundary; this boundary consists of two geodesics bounding flat half-neighborhoods. It is easy to show from the homogeneity of the flat metrics near such geodesics that when any two such half-neighborhoods are identified along the geodesic, the metric extends smoothly across the seam: the Riemannian manifold $(M_r, \sigma_0|dz|^2)$ is defined by identifying the curves $[0, 1)\times\{r\}$ in $(M, \sigma_0|dz|^2; p) \sim$ (the two copies of P_r) by $(x, r) \sim (x, r)$ and extending $\sigma_0|dz|^2$ across the geodesic seam $[0, 1)\times\{r\}$. We note that (i) there exists a canonical conformal embedding $i_r\colon (M_r \sim ([0, 1)\times\{r\}), \sigma_0|dz|^2) \to (M, \sigma|dz|^2; p)$, (ii) as representatives of a path in the compactified moduli space of complex structures $\overline{\mathscr{M}}_g$, $(M_r, \sigma_0|dz|^2)$ converges to $(M, \sigma|dz|^2; p)$, and (iii) recalling that only the complex structure of the source is relevant for the construction of harmonic maps, there exists unique harmonic diffeomorphisms from $(M_r, \sigma_0|dz|^2)$ to $(N, \rho|dw|^2)$ in each homotopy class.

Let M_r^c denote $i_r(M_r \sim ([0, 1] \times \{r\}))$. In what follows, we will often leap from M_r to $M_r^c \subset M$ and back. We now specify the homotopy class of maps that will interest us. Consider a diffeomorphism $h_1\colon M_1 \to N$ for which $h_1 \circ i_1^{-1}(\partial M_1^c)$ is the geodesic γ on (N, ρ). Then consider M_1^c as embedded in M_r via $i_r^{-1} \circ i_1$; we can then define a map $h_r\colon M_r \to N$ by requiring that $h_r = h_1$ on M_1 and the map h_r must satisfy $h_r(x, y) = h_1(x, 1)$ on $M_r \sim M_1^c \cong [0, 1) \times [1, 2r - 1]$, projecting the cylinder $M_r \sim M_1^c$ directly onto the geodesic γ without any twisting.

We have just used the somewhat unnatural but convenient parametrization of $M_r \sim M_1^c$ as $[0, 1) \times [1, 2r - 1]$ (with identifications) for the first of many times. The reader should notice that in this parametrization, if we describe $P_r \sim P_1$ as $P_r \sim P_1 = \{(x, y)|x \in [1, 0), 1 \le y \le r\}$ (with identifications), then the two copies of $P_r \sim P_1$ embed in $M_r \sim M_1^c = [0, 1) \times [1, 2r - 1]$ by the two maps (i) $(x, y) \mapsto (x, y)$ and (ii) $(x, y) \mapsto (1 - x, 2r - y)$. Thus, in the second map, $P_r - P_1$ is embedded backwards and upside down in $[0, 1) \times [1, 2r - 1]$ with ∂P_1 identified setwise with $[0, 1) \times \{1, 2r - 1\}$.

We may now define $w_r\colon (M_r, \sigma_0|dz|^2) \to (N, \rho|dw|^2)$ to be the unique harmonic diffeomorphism homotopic to h.

Proposition 3.1. *A subsequence of the maps w_s converges to a harmonic map $w\colon (M, \sigma|dz|^2; p) \to (N, \rho|dw|^2)$, with uniform convergence on compacta of $M \sim p$.*

Proof. Let $e(w_s)$ denote the energy density $e(w_s; \sigma_s, \rho)$. Then for a Lipschitz map $f_s\colon M_s \to N$ which is homotopic to $w_s\colon M_s \to N$, the harmonicity of w_s ensures that

$$\int_{M_s} e(w_s)\, dA(\sigma_0) \le \int_{M_s} e(f_s)\, dA(\sigma_0).$$

We consider a convenient candidate f_s. First we note that M_r^c has two boundary components; again let γ denote the geodesic on N homotopic to $w_r(\partial M_1^c)$. Let φ be a constant speed parametrization of γ so that $\varphi\colon [0, 1] \to \gamma$ with $\varphi(0) = \varphi(1)$. Consequently we can consider the harmonic maps Dirichlet problem of finding a harmonic map

$$\alpha_r\colon (M_r^c; \partial_1 M_r^c, \partial_2 M_r^c) \to (N - \gamma, \gamma, \gamma)$$

so that $\alpha_r = \varphi$ on $\partial_i M_r^c$ (here we consider φ as defining a map $\varphi\colon \partial_i M_r^c = [0, 1) \times \{r\} \to \gamma$). The theory of Hamilton [10] tells us that there exists a unique such map α_r, since $\partial(N - \gamma)$ is geodesic.

Next, on the pair of cylinders $M_s^c - M_r^c$ with $s > r$, we define a map β_{s-r} so that $\beta_{s-r}(x, y) = \varphi(x)$. We claim the following lemma.

Lemma 3.2.

$$\iint_{M_s^c - M_r^c} e(w_s)\, dA \geq \iint_{M_s^c - M_r^c} e(\beta_{s-r})\, dA \quad \textit{for } s > r.$$

Proof of Lemma 3.2.

$$\begin{aligned}
\iint_{M_s^c - M_r^c} e(w_s)\, dA &= 2\int_r^s \int_0^1 \frac{1}{2}\left[\left\| w_s^* \frac{\partial}{\partial x}\right\|_\rho^2 + \left\| w_s^* \frac{\partial}{\partial y}\right\|_\rho^2\right] dx\, dy \\
&\geq \int_r^s \int_0^1 \left\| w_s^* \frac{\partial}{\partial x}\right\|_\rho^2 dx\, dy \\
&\geq \int_r^s \left[\int_0^1 \left\| w_s^* \frac{\partial}{\partial x}\right\|_\rho dx\right]^2 dy \\
&\geq (s - r)(\text{minimum } \rho\text{-length of curve homotopic to } w_s([0, 1) \times \{y\}))^2 \\
&= \iint_{M_s^c - M_r^c} e(\beta_{s-r})\, dA. \quad \text{q.e.d.}
\end{aligned}$$

We now continue with the proof of Proposition 3.1. Set

$$f_s(q) = \begin{cases} \alpha_r(q) & \text{if } q \in M_r^c, \\ \beta_{s-r}(q) & \text{if } q \in M_s^c \sim M_r^c. \end{cases}$$

Then, because w_s is energy minimizing,

$$\begin{aligned}\iint_{M_r^c} e(w_s)\,dA(\sigma_0) &+ \iint_{M_s^c - M_r^c} e(w_s)\,dA(\sigma_0)\\ &= \iint_{M_s} e(w_s)\,dA(\sigma_0)\\ &\leq \iint_{M_s} e(f_s)\,dA(\sigma_0)\\ &= \iint_{M_r^c} e(\alpha_r)\,dA(\sigma_0) + \iint_{M_s^c - M_r^c} e(\beta_{s-r})\,dA(\sigma_0)\\ &\leq \int_{M-r^c} e(\alpha_r)\,dA(\sigma_0) + \int_{M_s^c - M_r^c} e(w_s)\,dA(\sigma_0)\end{aligned}$$

by Lemma 3.2. So

$$\iint_{M_r^c} e(w_s)\,dA(\sigma_0) \leq \iint_{m_r^c} e(\alpha_r)\,dA(\sigma_0) = C(M_r^c) \quad \text{for all } s > r. \tag{3.0}$$

Standard estimates can now be used to complete the proof of Proposition 3.1. (See, for example, [18]: from the Eells-Sampson [7] estimate $\Delta_{\sigma_r} e(w_s) \geq 2(\inf K_{\sigma_r}) e(s_w)$ valid for a harmonic map w_s on (M_r^c, σ_r), we derive the estimate (see [7] or [18, p. 336]) valid for $q \in M_r^c$,

$$e(w_s)(q) \leq C(M_r) \iint_{M_r^c} e(w_s)\,dA.$$

Thus, by (3.0) we conclude that

$$e(w_s)(q) \leq C(M_r) \iint_{M_r^c} e(\alpha_r)\,dA = C(M_r^c) \quad \text{for all } s > r.$$

This then shows that the right-hand side of the harmonic map equation, $\Delta_s w_s = (\log \rho)_{w_s} (w_s)_z (w_s)_{\bar{z}} / \sigma(z)$, is bounded, giving a $C^{1,\alpha}$ estimate on w_s by standard potential theory arguments, which in turn gives a C^α estimate on the right-hand side. We repeat the process to get a $C^{2,\alpha}$ estimate on w_s, and then use Ascoli-Arzela to find the convergence of a subsequence of the $\{w_s\}$ to a harmonic map on M_r^c).

A diagonal process then gives convergence of another subsequence of the w_s to a harmonic map w on $M \sim p$, with uniform convergence on compacta of $M \sim p$. q.e.d.

Next we consider an example of this convergence: we consider the asymptotics of maps between cylinders, where the source cylinder "develops a node" and the target cylinder remains fixed. In particular, consider the boundary value problem of harmonically mapping the cylinder

$M = [0, 1] \times [1, 2s-1]$ (with boundary identifications $\{0\} \times [1, 2s-1] \cong \{1\} \times [1, 2s-1]$), to the cylinder $N = [l^{-1}\csc^{-1} l^{-1}, \pi/l - l^{-1}\csc^{-1} l^{-1}] \times [0, 1]$, with boundary identifications

$$[l^{-1}\csc^{-1} l^{-1}, \pi/l - l^{-1} csc^{-1} l^{-1}] \times \{0\}$$
$$\cong [l^{-1}\csc^{-1} l^{-1}, \pi/l - l^{-1}\csc^{-1} l^{-1}] \times \{1\},$$

where we require $w(z) = u(x, y) + iv(x, y)$ take $[0, 1] \times \{1\}$ to $\{l^{-1}\csc^{-1} l^{-1}\} \times [0, 1]$ and $[0, 1] \times \{2s-1\}$ to $\{\pi/l - l^{-1}\csc^{-1} l^{-1}\} \times [0, 1]$. We choose a conformal structure determined by a flat metric $|dz|^2$ on M, and we choose a hyperbolic metric $l^2 \csc^2 lu|dw|^2$ on N, where the core geodesic $\{\pi/2l\} \times [0, 1]$ has length l.

We first observe that both M and N admit an anti-isometric reflection about the curves $[0, 1] \times \{s\}$ and $\{\pi/2l\} \times [0, 1]$. Thus it suffices to solve the problem corresponding to the one above with $M = [0, 1] \times [1, s]$ and $N = [l^{-1}\csc^{-1} l^{-1}, \pi/2l] \times [0, 1]$. We see that $v(x, y) = x$ and $u(x, y) = u(y)$ solve the Euler-Lagrange equation $u'' = l(\cot lu)(u'^2 - 1)$. It is not hard to show that for s sufficiently large, the solution u_s is concave, and indeed converges to a solution u to the "noded" problem, i.e., $M = [0, 1] \times [1, \infty)$, where we require $u(1) = l^{-1}\cosh^{-1} l^{-1}$ and $\lim_{y\to\infty} u(y) = \pi/2l$.

This "noded" problem has the explicit solution

$$u(y) = \frac{1}{l}\sin^{-1}\left\{\frac{1 - e^{2l(1-y)}[(1-l)/(1+l)]}{1 + e^{2l(1-y)}[(1-l)/(1+l)]}\right\}. \tag{3.1}$$

If the boundary curve $(u(1), v)$ had length λ, the factor $(1-l)/(1+l)$ would be replaced by $(\sqrt{\lambda} - l)/(\sqrt{\lambda} + l)$.

From an argument like that for Lemma 3.2, we can easily surmise that the energy of the maps w_s constructed for Proposition 3.1 must grow arbitrarily large. Thus, if we expect a limiting map to be a smooth diffeomorphism from $M \sim p$ to $N \sim \gamma$ as in (3.1), we must expect the map to have infinite energy. Yet this infinitude will arise only because of the neighborhood of the node: $(M, \sigma|dz|^2; p)$ has vanishing injectivity radius, or in the flattened metric, $(M, \sigma_0|dz|^2; p)$ has infinite area. Thus we might expect the map in a neighborhood of the node to be a perturbation of (3.1). We aim to prove this by comparing the two maps. We begin first with

Lemma 3.3. *Let (C, ρ) be a bi-infinite hyperbolic cylinder with core geodesic $\overline{\gamma}$, i.e., C is isometric to $\mathbf{H}^2/\langle z \mapsto \lambda^2 z\rangle$, where $\mathbf{H}^2$ is the upper half-plane $\{\operatorname{Im} z > 0\}$ endowed with the hyperbolic metric $|dz|^2/y^2$. Then*

$d_\rho^2(\bar\gamma, \cdot)$ is a convex C^2 function on C, and if $u\colon (\Omega, g) \to (C, \rho)$ is a harmonic map from a domain Ω into C, then

$$\begin{aligned}(3.2)\qquad &\Delta_g d_\rho^2(u(z), \bar\gamma)\\ &\quad\geq 2\min(1, d_\rho(u(z), \bar\gamma)\tanh d_\rho(u(z), \bar\gamma))e(u(z); g, \rho).\end{aligned}$$

Proof. It is well known (e.g. see [14, equation (5.1.1)]) that if $f \in C^2(C, \mathbf{R})$, u is harmonic, and $\{e_\alpha\}$ is frame on Ω, then

$$\Delta_g f \circ u = \sum_\alpha \operatorname{Hess} f(u_* e_\alpha, u_* e_\alpha).$$

Now use as a fundamental domain for C in $\mathbf{H}^2$ the region in the upper half-plane between $|z| = 1$ and $|z| = \lambda^2$; then in the orthonormal coordinates

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\right\} = \{\xi_1, \xi_2\},$$

which are perpendicular and tangent to the level sets of $d_\rho^2(\bar\gamma, \cdot)$, one easily computes

$$\operatorname{Hess} d_\rho^2(\bar\gamma, c) = 2\xi_1 \otimes \xi_1 + 2\, d_\rho(\bar\gamma)\tanh d_\rho(\bar\gamma, \cdot)\xi_2 \otimes \xi_2.$$

From $e = \frac{1}{2}\sum_\alpha \|u_* e_\alpha\|_\rho^2$, (3.2) follows. q.e.d.

Recall the harmonic map $w\colon (M, \sigma|dz|^2, p) \to (N, \rho|dw|^2)$ constructed in Proposition 3.1.

Proposition 3.4. *$w\colon (M, \sigma|dz|^2, p) \to (N, \rho|dw|^2)$ is a diffeomorphism between $M \sim p$ and $N \sim \gamma$.*

Proof. Consider half-collar neighborhoods $\mathscr{C}_i$ in M_r^c of the boundaries $\partial_i M_r^c$. Since, from (3.0), $\int_{M_r^c} e(w_s)\, dA(\sigma_0) < C(M_r^c)$, we know that $\int_{\mathscr{C}_i} e(w_s)\, dA(\sigma_0) < C(M_r^c)$ for $s > r$. By the Courant-Lebesgue Lemma (see [14, Lemma 3.1, p. 20]), we conclude that there are curves $c_{s,i} \subset \mathscr{C}_i$, so that the ρ-lengths of their images $l_\rho(w_s(c_{s,i})) < K_1(r)$. Consider a covering space $\overline{N} \to N$ of N corresponding to the homotopy class $[\gamma] \in \pi_1 N$ of $\gamma \subset N$; we represent $\overline{N}$ as a quotient of the hyperbolic upper half-plane by the group generated by an isometry $\alpha\colon z \mapsto \lambda^2 z$ for some λ. Then $\overline{N}$ is geometrically a bi-infinite cylinder with core geodesic $\bar\gamma$, and $\bar\gamma$ projects to $\gamma \subset N$. The harmonic map w_s, then restricted to the cylinder $(M_s \sim M_r^c) \cup \mathscr{C}_i$, lifts to a map $\overline{w}_s\colon (M_s \sim M_r^c) \cup \mathscr{C}_i \to \overline{N}$. Because $l_\rho(w_s(c_{s,i})) < K_1(r)$, we have $d_{\overline{N}}(q_i, \bar\gamma) < K_2(r)$ for $q_i \in \overline{w}_s(c_{s,i})$, and so for $q_i' \in \overline{w}_s(\partial_i M_r^c)$, $d_{\overline{N}}(q_i', \bar\gamma) < K_3(r)$, an estimate which is independent of s.

We now want to apply Lemma 3.3 to the harmonic map $\overline{w}_s : (M_s \sim M_r^c, \sigma_0) \to (\overline{N}, \rho)$ to conclude that $\overline{w}_x(\partial_i M_R^c)$ can be made arbitrarily close to $\overline{\gamma}$ by choosing s and R sufficiently large. We first notice that the lemma implies that $d_\rho^2(\overline{w}_s(z), \overline{\gamma})$ is subharmonic and consequently $d_\rho^2(\overline{w}_s(z), \overline{\gamma}) < K_3(r)^2$ for all $z \in M_s$. Next we need a crude estimate on $e(\overline{w}_s(z); \sigma_0, \rho)$. The maximum principle applied to equation (2.2) shows that for maps between closed hyperbolic surfaces, $\mathscr{H}(w_s; \sigma, \rho) \geq 1$; thus $e(\overline{w}_s; \sigma_0, \rho) = (\sigma/\sigma_0)e(w_s; \sigma, \rho) \geq \sigma/\sigma_0 = 1/\lambda$ so that $e(\overline{w}_s; \sigma_0, \rho) \geq 1/y^2$ in the coordinates on $P_r \sim P_s$. Identify $M_s \sim M_r^c$ with $[0, 1) \times [r, 2s - r]$. We want to compare $d_\rho^2(\overline{w}_s(z), \overline{\gamma})$, which satisfies inequality (3.2) on $M_s \sim M_r^c = [0, 1) \times [r, 2s - r]$, with a function $d_s(z) = d_s(y)$, a function of y only, defined on the domain $[0, 1) \times [r, 2s - r]$ which satisfies the ordinary differential equation

$$(3.3) \quad d_s''(y) = \min\left(1, \sqrt{d_s(y)} \tanh \sqrt{d_s(y)}\right) \begin{Bmatrix} 1/y^2 & \text{if } y < s, \\ 1/(2s - y)^2 & \text{if } y \geq s \end{Bmatrix}$$

with boundary conditions $d_s(r) = d_s(2s - r) = K_3(r)^2$. To compare $d_\rho^2(\overline{w}_s(z), \overline{\gamma})$ with $d_s(z)$ on $M_s \sim M_r^c \cong [0, 1) \times [r, 2s - r]$, we notice that our estimates on $d_\rho^2(\overline{w}_s(z), \overline{\gamma})$ imply that $d_\rho^2(\overline{w}_s(z), \overline{\gamma}) = d_s(z) < 0$ on $\partial(M_s \sim M_r^c) \cong [0, 1) \times \{r, 2s - r\}$; thus, if there is a point $z \in M_s \sim M_r^c$ at which $d_\rho^2(\overline{w}_s(z), \overline{\gamma}) - d_s(z)$ is positive, it is an interior point of $M_s \sim M_r^c$.

Therefore at a positive interior maximum z_0 of $d_\rho^2(\overline{w}_s(z), \overline{\gamma}) - d_s(z)$, we must have

$$\Delta[d_\rho^2(\overline{w}_s(z), \overline{\gamma}) - d_s(z)] \leq 0;$$

however, by comparing the right-hand sides of (3.2) and (3.3), we find that if $d_\rho^2(w_s(z_0), \overline{\gamma}) > d_s(z_0)$, then $\Delta[d_\rho^2(\overline{w}_s(\cdot), \overline{\gamma}) - d_s(\cdot)] > 0$ at z_0. Hence $d_\rho^2(\overline{w}_s(z)) \leq d_s(z)$ for $z \in M_s \sim M_r^c$.

We are left to investigate the ordinary differential equation (3.3) and its solution $d_s(z)$; we claim that for all ε, there is an R_0 and an S_0 depending on R_0, so that for $R > R_0$ and $s > S_0$, we have $d_s(R) < \varepsilon$. As a consequence of the claim we would then have that $\sup_{z \in \partial_i M_R} d_\rho^2(\overline{w}_s(z), \overline{\gamma}) < \varepsilon$. Moreover, because $d_\rho^2(\overline{w}_s(z), \overline{\gamma})$ is subharmonic, we would also conclude that $\sup_{z \in \partial_i M_s} d_\rho^2(\overline{w}_s(z), \overline{\gamma}) < \varepsilon$.

To establish the claim, we first show that for all $\varepsilon > 0$, there is an S_0 so that for $s > S_0$, $d_s(s) < \varepsilon$. To see this, we observe that the solution $d_s(y)$ of (3.3) is symmetric about $y = s$, with $d_s'(s) = 0$ and

$d_s(y) \geq d_s(s)$. Thus, if for all s, $d_s(s) \geq \varepsilon$ for some $\varepsilon > 0$, then on the interval $[s, 2s-r]$, $d_s(y)$ would satisfy

$$d_s(y) \geq \varepsilon, \qquad d_s(2s-r) = k_3(r)^2,$$
$$d_s'(s) = 0, \qquad d_s''(y) \geq [\varepsilon^{1/2}\tanh(\varepsilon^{1/2})]/(2s-y)^2.$$

By integrating the last inequality twice we reach a contradiction, showing that for a given $\varepsilon > 0$, when s is sufficiently large we will have $d_s(s) > \varepsilon$. Finally choose $\varepsilon > 0$ and R_0 so that $d_{R_0}(R_0) < \varepsilon$. We observe that the strong maximum principle implies that if $s_2 > s_1$, then $d_{s_2}(y) < d_{s_1}(y)$.

Thus for all $s > R_0$, $d_s(R_0) < \varepsilon$, concluding the proof of the claim.

Next we argue that the limiting map w is a diffeomorphism. Since the Jacobian of the maps w_s satisfies $\mathscr{J}(w_s) > 0$, we conclude that $\mathscr{J}(w) \geq 0$. We follow an argument of Schoen-Yau [19, Proposition 2.2]. Suppose $\mathscr{J}(w)(q) = 0$, where $q \in M_{s_0}^c$.

We saw earlier that $\mathscr{H}(w_s; \sigma_0, \rho)(q) \geq 1/\lambda(z(q)) > 0$, and consequently $\mathscr{H}(w; \sigma_0, \rho)(q) \geq 1/\lambda(z(q))$. But since $\mathscr{J}(w)(q) = 0$, we also have that $\mathscr{L}(w; \sigma_0, \rho)(q) = \mathscr{H}(w; \sigma_0, \rho)(q) > 0$. Next we use the equation

$$\Delta_{\sigma_0} \log \frac{\mathscr{H}}{\mathscr{L}} = -4K(p)\mathscr{J} = +4\mathscr{J}, \tag{3.4}$$

which is also derived in [19], and we consider a neighborhood V around q where both $\mathscr{L}(w)$ and $\mathscr{H}(w)$ are both strictly positive. In this neighborhood, we have that

$$4\mathscr{J} = 4(\mathscr{H} - \mathscr{L}) \leq c_1\left(\frac{\mathscr{H}}{\mathscr{L}} - 1\right) \leq c_2 \log\frac{\mathscr{H}}{\mathscr{L}}.$$

Setting $k = \log(\mathscr{H}/\mathscr{L})$, we see that (3.4) yields $\Delta_{\sigma_0} k \leq c_2 k$ in V. Thus Lemma $6'$ of [12] gives

$$\iint_{|\zeta|\leq R} k(\zeta)\, d\xi\, d\eta \leq c_3 k(0), \tag{3.5}$$

where $\zeta = \xi + i\eta$ is a flat σ_0-isothermal coordinate system around q. Since $k(0) = \log(\mathscr{H}(w)(q)/\mathscr{L}(w)(q)) = 0$ by hypothesis, and $\mathscr{J} \geq 0$ in V implies that $k \geq 0$ in V, we see from (3.5) that $k \equiv 0$ in a neighborhood of q.

Thus if $\mathscr{J}(w)$ has zeros, $\mathscr{J}(w)$ must be identically zero. This we rule out from the previous geometric argument. We choose R and S_0 as before so that $\sup_{z\in\partial_i M_R} d_\rho^2(w_s(z), \gamma) < \varepsilon$ (here we can work on N

instead of the cover $\overline{N}$ since we are in a small neighborhood of $\overline{\gamma}$ and the covering map is a local homeomorphism). Then

$$\iint_{M_R} \mathscr{J}(w_s)\, dA(\sigma_0) > A(N) - c(\varepsilon),$$

where $c(\varepsilon)$ is a small constant depending only on ε. Thus it is impossible that $\mathscr{J}(w) = \lim \mathscr{J}(w_s)$ vanishes identically on M_R^c (or on M), and so w has positive Jacobian everywhere. Since on each M_R^c, w is the limit of diffeomorphisms with Jacobian uniformly bounded away from zero, w is a diffeomorphism of M_R^c, with $N \sim w(M_R^c)$ isotopic to a neighborhood of the geodesic γ. Moreover, because we can force $w_s(\partial M_R)$ to be arbitrarily close to γ by choosing R and S sufficiently large, we know that we can force $w(\partial M_R)$ to be arbitrarily close to γ by choosing R sufficiently large. Thus w takes $M \sim p$ onto $N \sim \gamma$.

Finally, suppose that there were a point q so that $w(q) \in \gamma$. Now a neighborhood of the node p has two components C_2 and C_2 in $M \sim p$, and we might as well assume that $q \in C_1$. But then a small neighborhood O of q is cut into two components by the pre-image of γ, say O_1 and O_2. But then there are points q_1 and q_2 in O_1 and O_2, respectively, that are mapped to different components of a neighborhood of γ in $N \sim \gamma$; further we suppose that $d_N(w(q_i), \gamma) = \varepsilon > 0$. However, our estimates show that we can find a neighborhood U of the node that does not include q_i and such that $U \cap C_1$ is mapped locally homeomorphically and incompressibly into an $\varepsilon/2$ neighborhood of γ. So $w(U \cap C_1)$ contains a curve homotopic to γ in N and within ρ-distance of $\varepsilon/2$ from γ. This curve, though, would intersect the image of any small arc in O which connected q_1 and q_2. This would contradict w being a diffeomorphism, so we conclude that w is a diffeomorphism between $M \sim p$ and $N \sim \gamma$. q.e.d.

Next we derive some properties of the diffeomorphism w. We first want to show that w is asymptotic to the map $u(z) = u(y)$ of (3.1). First we need a lemma like Lemma 3.3, but before we can state it, we need some preparations.

Consider two harmonic maps u_1, $u_2 \colon \Omega \to N$ of class $C^0(\overline{\Omega}, N) \cap C^2(\Omega, N)$, where Ω is a domain and $u_i(\overline{\Omega}) \subset B_R(p)$ is a ball of ρ-radius R in N which is disjoint from the cut locus of p. Define

$$Q(z) = \cosh d_\rho(u_1(z), u_2(z)) - 1;$$

$Q(z)$ is smooth and well defined because of cut-locus restrictions on the images of Ω under u_i. Suppose that $d_\rho(u_1(z), u_2(z)) \neq 0$. Then let e_α

be a tangent vector of Ω at z, and let

$$\mathrm{Tan}(u_{1*}e_\alpha) = \langle u_{1*}e_\alpha, \text{grad } d_\rho(u_2(z), \cdot)|_{u_1(z)}\rangle_\rho$$

and

$$\mathrm{Nor}(u_{1*}e_\alpha) = u_{1*}e_\alpha - \mathrm{Tan}(u_{1*}e_\alpha)(\text{grad } d(u_2(z), \cdot)|_{u_1(z)}),$$

with a similar definition for $u_{2*}e_\alpha$. Here $\mathrm{Tan}\, u_{i*}e_\alpha$ is a number, and $\mathrm{Nor}\, u_{i*}e_\alpha$ is a vector. If $d_\rho(u_1(z), u_2(z)) = 0$, then set $\mathrm{Tan}\, u_{i*}e_\alpha = 0$ and $\mathrm{Nor}\, u_{1*}e_\alpha = 0$; this case will be of little importance to us. Let $\{e_\alpha(z)\}$ be an orthogonal frame at z.

Lemma 3.5 (see [13]). *Using the hypotheses and the notation as above, Q satisfies*

$$\Delta Q(z) \le (Q(z)+1)\left(\sum_{\alpha=1}^{2}\left(\sum_{i=1}^{2}\mathrm{Tan}\, u_{i*}e_\alpha\right)^2\right) + Q(z)\sum_{\alpha=1}^{2}\left(\sum_{i=1}^{2}\|\mathrm{Nor}\, u_{i*}e_\alpha\|_\rho^2\right). \tag{3.6}$$

Proof. This is a simple extension to negative curvature of the result in [13] that $[d_\rho(u_1(z), u_2(z))]^2$ is subharmonic under the above hypothesis with N nonpositively curved. (See [14, Theorem 5.1, pp. 54–60]). q.e.d.

Remark. The right side of (3.6) is nonnegative, but could vanish, if, for instance, u_i mapped Ω into a geodesic segment, the two maps being a constant distance apart; in that case, both $\|\mathrm{Nor}\, u_{i*}e_\alpha\|^2$ and $\sum_{i=1}^2 \mathrm{Tan}\, u_{i*}e_\alpha$ vanish because $u_{i*}e_\alpha$ is entirely along the geodesic connecting $u_i(z)$, with equivalent parametrizations.

We now begin our comparison of w to the map $u(z)$ in the domain $M \sim M_1^c$. Let $r = 1$ and consider the set of cylinders $M_s \sim M_1^c$, parametrized as usual by $[0, 1) \times [1, 2s-1]$ with the usual identifications. The map w takes $M \sim M_1^c$ into a neighborhood U of $\gamma \subset N$, and we choose coordinates on U so that

$$U = [(\csc^{-1} l^{-1})/l, \pi/l - (\csc^{-1} l^{-1})/l] \times [0, 1)$$

with metric $\rho = l^2 \csc^2 lu(du^2 + dv^2)$ with the horizontal boundary lines identified; here γ is $\{\pi/2l\} \times [0, 1)$.

Consider a map $f_s : M_s \to N$ so that (i) f_s is homotopic to w_s, and (ii) $f_s(M_s \sim M_1^c) \subset U$ with

$$f_s(x, 1) = (\csc^{-1} l^{-1}/l, x), \quad f_x(x, 2s-1) = (\pi/l - (\csc^{-1} l^{-1})/l, x),$$

and before, $u_s(u)$ converges to $u(y)$ as defined by (3.1). Let $f\colon M \sim p \to N \sim \gamma$ be the limit of the maps $f_s\colon M_s^c \to N$ as $s \to \infty$. So the maps f_s between surfaces extend the harmonic maps $(u_s(y), x)$ between cylinders.

We want to compare w_s and f_s on $M_s \sim M_1^c$; to do this, we need to use a technique of Schoen and Yau [20] that will adapt Lemma 3.5 to the situation where the domain is not simply connected.

Let $F_s\colon (M_s \sim M_1^c) \times [0, 1) \to N$ be a homotopy between $F_s(z, 0) = w$ and $F_s(z, 1) = f_s$. We lift F_s to $\tilde{F}_s\colon (M_s \sim M_1^c)^\sim \times [0, 1] \to \tilde{U}$ and obtain liftings $\tilde{w}_s = \tilde{F}_s(\tilde{z}, 0)$ and $\tilde{f}_s = \tilde{F}_s(\tilde{z}, 1)$. We notice that, because of the explicit description of f_s, we have that if M^* is a particular representative for $M_s \sim M_1^c$ in $(M_s \sim M_1^c)^\sim$, then for any pair of points z^*, $z_0^* \in M^*$, $d_{\tilde{U}}(\tilde{f}_s(z^*), \tilde{f}_s(z_0^*)) < k_0$, independent of s. Now, on $\tilde{U} \times \tilde{U}$, $\pi_1 U$ acts by isometries and $\cosh \tilde{d}_{\tilde{U}\times\tilde{U}} - 1$ is a C^2 function. Moreover, since for $a \in \pi_1(M_s \sim M_1^c)$, there is an $\alpha \in \pi_1 U$ with $\tilde{w}_s(a(\tilde{z})) = \alpha\tilde{w}_s(\tilde{z})$ and $\tilde{f}_s(a(\tilde{z})) = \alpha\tilde{f}_s(\tilde{z})$, we see that if we define $\tilde{g}_s = (\tilde{w}_s, \tilde{f}_s)\colon (M_s \sim M_1^c)^\sim \to \tilde{U} \times \tilde{U}$, then $\tilde{g}_s$ is harmonic and so induces a harmonic map $g_s\colon M_s \sim M_1^c \to (\tilde{U} \times \tilde{U})/(\pi_1 U)$. Then we extend our construction of Lemma 3.5 to the nonsimply connected $M_s \sim M_1^c$: the function $\overline{Q}_s = \cosh \overline{d}_{(\tilde{U}\times\tilde{U})/(\pi_1 U)} \circ g_s - 1$ is C^2 on $M_s \sim M_1^c$ and satisfies inequality (3.6) in place of Q.

We want to bound the boundary conditions of $\overline{Q}_s$ on $M_s \sim M_1^c$. Denote by $\partial_1 M_1^c$ and $\partial_2 M_1^c$ the two boundary components of ∂M_1^c, and recall that $w_s(\partial M_1^c)$ converges. Then from our choice of homotopy class of w_s, we see that for fixed $\tilde{z}_1 \in (M_s \sim M_1^c)^\sim$ above $z_1 \in \partial_1 M_1^c$ and fixed $\tilde{z}_2 \in (M_s \sim M_1^c)^\sim$ above $z_2 \in \partial_2 M_1^c$, we have $d_{\tilde{U}}(\tilde{w}_s(\tilde{z}_1), \tilde{w}_s(\tilde{z}_2)) < k_1$, where k_1 is independent of s. Thus, since the same is certainly true by construction for $\tilde{f}_s$, the lift of f_s determined by F_s, we see that for $\tilde{z} \in (M_s \sim M_1^c)^\sim$ above $z \in \partial M_1^c$, $d_{\tilde{U}}(\tilde{w}_s(\tilde{z}), \tilde{f}_s(\tilde{z})) < k_2$ where k_2 is independent of s. We conclude that $\overline{Q}_s$ is bounded on ∂M_1^c by k_3, independently of s. Since the right-hand side of (3.6) is nonnegative, we find that $\overline{Q}_s$ is bounded on all $M_s \sim M_1^c$, independently of s.

For each s, $M_s^c \sim M_1^c$ is a pair of cylinders; we now want to lift all of the pairs of cylinders $\{M_s^c \sim M_1^c | s > 1\}$ as well as $M \sim p \sim M_1^c$ in a consistent way so that we may obtain well-defined limiting maps $\tilde{w}$, $\tilde{f}\colon (M \sim p \sim M_1^c)^\sim \to \tilde{U}$ which cover w, $f\colon M \sim p \sim M_1^c \to U$, and which satisfy $d_{\tilde{U}}(\tilde{w}, \tilde{f}) < k_2$. So define covers $(M_s^c \sim M_1^c)^\sim$ and

$(M \sim p \sim M_1^c)^\sim$ of $M_s^c \sim M_1^c$ and $M \sim p \sim M_1^c$, respectively, and lifts $\tilde{w}_s$, $\tilde{f}_s\colon (M_s^c \sim M_1^c)^\sim \to \tilde{U}$ of w_s, $f_s\colon M_s^c \sim M_1^c \to U$ so that : (i) $(M_s^c \sim M_1^c)^\sim$ embeds in $(M \sim p \sim M_1^c)^\sim$, (ii) for a pair of fixed points $\tilde{z}_1$ and $\tilde{z}_2$ above $z_1 \in \partial_1 M_1^c$ and $z_2 \in \partial_2 M_1^c$, respectively, and s, $s' > 1$

$$d_{\tilde{U}}(\tilde{w}_s(\tilde{z}_1), w_{s'}(\tilde{z}_1)) + d_{\tilde{U}}(\tilde{w}_s(\tilde{z}_2), w_{s'}(\tilde{z}_2)) < k_2,$$

(iii) as before, $d_{\tilde{U}}(\tilde{f}_s(\tilde{z}), \tilde{f}_s(\tilde{z}_0)) < k_0$ on any particular representative $M_s^c \sim M_1^c$ in $(M_s^c \sim M_1^c)^\sim$, for all $s > 1$, and (iv) for $\tilde{z}$ above $z \in M_r^c \sim M_1^c$ and for $s > 1$, $d_{\tilde{U}}(\tilde{f}_s(\tilde{z}), \tilde{w}_s(\tilde{(z)})) < k_2$, independently of s.

The last condition may be imposed because of the estimate $d_{\tilde{U}\times\tilde{U}/\pi_1 U} \circ g_s < k_2$, independently of s. We then let $s \to \infty$ and take subsequences as before to obtain limits $\tilde{w}$, $\tilde{f}\colon (M \sim p \sim M_1^c)^\sim \to \tilde{U}$ of $\tilde{w}_s$, $\tilde{f}_s\colon (M_s^c \sim M_1^c)^\sim \to \tilde{U}$; these maps $\tilde{w}$, $\tilde{f}\colon (M \sim p \sim M_1^c)^\sim \to \tilde{U}$ cover w, $f\colon (M \sim p \sim M_1^c) \to U$. Furthermore, the map $\tilde{g} = (\tilde{w}, \tilde{f})\colon (M \sim p \sim M_1^c)^\sim \to \tilde{U} \times \tilde{U}$ induces a map $g\colon (M \sim p \sim M_1^c) \to \tilde{U} \times \tilde{U}/\pi_1 U$, and our estimate $\overline{Q}_s < k_3$, together with the estimates from (ii) and (iii) of our construction, implies that $\overline{Q} = d_{\tilde{U}\times\tilde{U}/\pi_1 U} \circ g$ satisfies $\overline{Q} < k_4$ on $M \sim p \sim M_1^c$.

We intend to get a better estimate, but we first obtain a consequence of our normalizations of homotopy classes of f_s and w_s. We claim that given a particular representative M^* of $(M \sim p \sim M_1^c)$ in $(M \sim p \sim M_1^c)^\sim$, for any pair of points z^*, $z_0^* \in M^*$, we have estimate $d_{\tilde{U}}(\tilde{w}(z^*), \tilde{w}(z_0^*)) < k_5$. This is an important point, for it says, informally, that the map w does not have any infinite 'twist', or equivalently, that the maps $w_s\colon M_2 \to N$ twist back and forth only a bounded amount. To see that claim, recall that we chose lifts $\tilde{f}_s$ of f_s so that $d_{\tilde{U}}(\tilde{f}_s(z^*), \tilde{f}_s(z_0^*)) < k_0$ for all s, hence $d_{\tilde{U}}(\tilde{f}(z^*), \tilde{f}(z_0^*)) < k_0$. Since we have just shown that $d_{\tilde{U}}(\tilde{f}(z^*), \tilde{w}(z)) < k_5$, the estimate follows.

To get a better estimate than $\overline{Q} < k_4$, we need to choose a better comparison map than the f we have been using; notice that there is an S^1 family of choices of boundary values for the comparison maps (3.1) to assume. We choose the correct boundary values in the following way.

Let C be a flat cylinder of length 1 obtained by our usual identifications of $[0, 1) \times [0, 1]$, let j_s^i $(i = 1, 2)$ be the conformal identification $j_s^i\colon C \to M_{s+1}^c \sim M_s^c$ into the two components of $M_{s+1}^c \sim M_s^c$, and let $\zeta_s^i\colon C \to N$ be the composition $w \circ j_s^i$. Then define a cover $\tilde{C} \cong \mathbf{R} \times [0, 1]$ of C and lifts $\tilde{j}_s^i = \tilde{C} \to (M \sim p \sim M_1^c)^\sim$ so that $\bigcup_s \tilde{j}_s^i(\{0\} \times [0, 1])$ is connected in $(M \sim p \sim M_1^c)^\sim$. Thus $\tilde{\zeta}_s^i = \tilde{w} \circ \tilde{j}_s^i = \widetilde{C} \to \widetilde{U}$ is a lift of ζ_s^i, and we observe from our argument of Proposition 3.4 that by choosing s large, we can force $\tilde{\zeta}_s^i$ to map arbitrarily closely to the lift $\tilde{\gamma}$ of the core geodesic γ. Now, as before, choose a particular representative M^* for $(M \sim p \sim M_1^c)$ in $(M \sim p \sim M_1^c)^\sim$, and suppose z^*, $z_0^* \in M^*$; by our construction of the lift $\tilde{j}_s^i : \widetilde{C} \to (M \sim p \sim M_1^c)^\sim$, we see that $(\tilde{j}_s^i)^{-1}(M^*)$ has compact closure. Since $d_{\widetilde{U}}(\tilde{w}(z^*), \tilde{w}(z_0^*)) < k_5$, we see that $d_{\widetilde{U}}(\tilde{\zeta}_s^i((\tilde{j}_s^i)^{-1} z^*), \tilde{\zeta}_s^i((\tilde{j}_s^i)^{-1} z_0^*))$ is bounded independently of s. This, together with (i) the bounded geometry of $\widetilde{U}$ and C, and (ii) the relative compactness of $(\tilde{j}_s^i)^{-1}(M^*)$, then implies that $e(\tilde{\zeta}_s^i; |dz|^2, \tilde{\rho})$ is bounded independently of s (cf. [14, Theorem 6.1, p. 72]). Consequently, by the equivariance of $\tilde{\zeta}_s^i$, $e(\zeta_s^i; |dz|^2, \rho)$ is bounded independently of s.

Thus by the same bootstrap argument as at the end of the proof of Proposition 3.1, a subsequence $\zeta_{s_k}^i$ of the harmonic maps ζ_s^i converges in $C^{2,\alpha}$ to a harmonic map $\zeta^i : (C, |dz|^2) \to \gamma$, here using that γ is the intersection of all arbitrary small neighborhoods of γ. Since the maps $\{\zeta_{s_k}^i\}$ are homeomorphisms of C, the limiting map ζ^i must be a degree one map onto γ; but the only such harmonic maps are the collapse maps β_1 constructed for Lemma 3.2 out of constant speed parametrizations $\varphi : [0, 1) \to \gamma$ of γ. Thus the limiting (as $s_k \to \infty$) image $w(\partial_i M_{s_k}^c)$ of the σ_0-geodesic $\partial M_{s_k}^c$ is a constant speed parametrization $\varphi : S^1 \to \gamma$ and

$$\lim e(w(\partial_i M_{s_k}^c); \sigma_0, \rho) = e(\beta_2; \sigma_0, \rho) = \tfrac{1}{2} l_\rho(\gamma)^2. \tag{3.7}$$

We notice that there is an S^1 family of such parametrizations.

Moreover, suppose that ζ_{s_k} converges to $\beta_1(\varphi)$ (here temporarily dropping the superscripts from ζ_s^i, so that we work in only one component $\mathscr{C}_1$ of $M \sim p \sim M_1^c$, and indicating the dependence of β_1 on the parametrization $\varphi : S \to \gamma$), but a different subsequence ζ_{s_l} converges to $\beta_1(\psi)$, where φ and ψ are different parametrizations; we claim that this leads to a contradiction. Let $s_k < s_l < s_K$ be points in the subsequences corresponding to φ, ψ, and φ respectively, and suppose first that each point is sufficiently large so that ζ_{s_k}, ζ_{s_l}, and ζ_{s_K} are much closer to $\beta_1(\varphi)$, $\beta_1(\psi)$, and $\beta_1(\varphi)$ in C^1 than φ is to ψ.

We also make another assumption, possibly requiring the passage to a further subsequence, to rid ourselves of a technical complication. Notice that on $\mathscr{C}_1 \cap (M_{s_K+1}^c \sim M_{s_k}^c)$, which we parametrize by $[0, 1) \times [s_k, s_K + 1]$, our first assumption guarantees that it is possible to join $w(x, s_k)$ and

$w(x, s_K + 1)$ by a path δ that is much smaller than the injectivity radius of N; our second assumption is that $w(\{x\} \times [s_k, s_K + 1]) \cup \delta$ is homotopically trivial, i.e., that it has winding number zero with respect to γ. We are permitted to make this assumption because we have shown that $d_{\widetilde{U}}(\tilde{w}(\tilde{z}^*), \tilde{w}(\tilde{z}_0^*)) < k_5$ for a representative M^* of $M \sim p \sim M_1^c$ in $(M \sim p \sim M_1^c)^\sim$ and $\tilde{z}^*$, $\tilde{z}_0^* \in M^*$: the winding number of $w(\{x\} \times [s_k, s_K + 1]) \cup \delta$ with respect to γ lies in a finite list. Our assumptions, then, contradict the harmonicity of w, since w restricted to $\mathscr{C}_1 \cap M^c_{s_K+1} \sim M^c_{s_k}$ must minimize energy among maps u with the boundary conditions

$$u|_{\partial_1 M_{s_k}} = w|_{\partial_1 M_{s_k}} \cong \varphi \quad \text{and} \quad u|_{\partial_1 M_{s_k+1}} = w|_{\partial_1 M_{s_k+1}} \cong \varphi, \tag{3.8}$$

where $\partial_1 M_r = \mathscr{C}_1 \cap \partial M_r$. Thus, if $w(\partial_1 M_{s_l}) \cong \psi$, an easy computation shows that we could lower the energy of w on $\mathscr{C}_1 \cap M^c_{s_K+1} \sim M^c_{s_k}$ by replacing the map there by a map which is very near $\beta_{s_K+1-s_k}(\varphi)$. Hence ζ_s is $C^{2,\alpha}$ close to $\beta_1(\varphi)$ for all s sufficiently large, without passage to a subsequence.

Suppose that on each component $\mathscr{C}_i$ of $M \sim p \sim M_1^c$, the maps $\zeta_s^i = w \circ j_s^i$ limit on $\beta_1(\varphi_i)$ as $s \to \infty$. Then choose f so that the compositions $f \circ j_s^i$ also limit on $\beta_1(\varphi_i)$ as $s \to \infty$. Thus on both components of $M \sim p \sim M_1^c$, we have

$$\lim_{\substack{z \to p \\ z \in M \sim p \sim M_1^c}} d_U(w(z), f(z)) = 0.$$

We can now compare w and our new map f in the region $M \sim p \sim M_1^c$. As before, we let $F: (M \sim p \sim M_1^c) \times [0, 1] \to U$ be a homotopy between w and f, and we lift F to $\widetilde{F}: (M \sim p \sim M_1^c)^\sim \times [0, 1] \to \widetilde{U}$, obtaining lifts $\tilde{w} = \widetilde{F}(z, 0)$ and $\tilde{f} = \widetilde{F}(z, 1)$. Then the map $\tilde{g} = (\tilde{w}, \tilde{f}): (M \sim p \sim M_1^c)^\sim \to \widetilde{U} \times \widetilde{U}$ is harmonic and induces a harmonic map $g: (M \sim p \sim M_1^c) \to \widetilde{U} \times \widetilde{U}/\pi_1 U$. Thus $\overline{Q} = \cosh d_{\widetilde{U} \times \widetilde{U}/\pi_1 U} \circ g - 1$ is C^2 on $M \sim p \sim M_1^c$, and we have already shown that $\overline{Q} < k_4$. Moreover, we have chosen f so that on both components of $M \sim p \sim M_1^c$, we have

$$\lim_{\substack{z \to p \\ z \in M \sim p \sim M_1^c}} d_U(w(z), f(z)) = 0.$$

We then conclude that $\lim_{z \to p, z \in M \sim p \sim M_1^c} \overline{Q}(z) = 0$.

We have concluded that in each component of $M \sim p \sim M_1^c$, the harmonic map w converges to $\beta_1(\varphi)$ in $C^{2,\alpha}$ (for some $\varphi: S^1 \to \gamma$)

as $y \to \infty$; here we adopt the notation that φ represents a pair of maps $\varphi_i: S^1 \to \gamma$, one in each component of $M \sim p \sim M_1^c$. This next result provides all of the technical estimates which we will need later.

Recall from the first paragraph of §2 the definitions

$$\mathscr{H}(\beta_1(\varphi); \sigma_0, \rho) = (\rho(\beta_1(\varphi))/\sigma_0(z)|(\beta_1(\varphi))_z|^2,$$
$$\mathscr{L}(\beta_1(\varphi); \sigma_0, \rho) = (\rho(\beta_1(\varphi))/\sigma_0(z))|(\beta_1(\varphi))_{\bar{z}}|^2,$$
$$e(\beta_1(\varphi); \sigma_0, \rho) = \mathscr{H} + \mathscr{L}, \quad \text{and} \quad \mathscr{J} = \mathscr{H} - \mathscr{L}.$$

We now define the Beltrami differential $\nu(w; \sigma_0, \rho)$ for a map

$$w: (M, \sigma_0|dz|^2) \to (M, \rho|dw|^2)$$

by $\nu(w; \sigma_0, \rho)(z) = w_{\bar{z}}(z)/w_z(z)$. Thus $\nu(w; \sigma_0, \rho)$ is a tensor depending on the choice of coordinates for its argument, and $|\nu(w; \sigma_0, \rho)|^2$ is a function, defined independently of coordinates. We compute that

$$\mathscr{H}(\beta_1(\varphi); \sigma_0, \rho) = \mathscr{L}(\beta_1(\varphi); \sigma_0, \rho) = l_\rho(\gamma)^2/4,$$
$$e(\beta_1(\varphi); \sigma_0, \rho) = l_\rho(\gamma)^2/2, \qquad \mathscr{J}(\beta_1(\varphi); \sigma_0, \rho) = 0.$$

Further, with respect to the coordinates z and w which we have been using on $M \sim p \sim M_1^c \cong [0, 1] \times [1, \infty)\}$ and U, we compute

$$\nu(\beta_1(\varphi); \sigma_0, \rho) = 1.$$

We conclude

Lemma 3.6. *For the harmonic diffeomorphism* $w: (M, \sigma_0|dz|^2; p) \to (N, \rho|dw|^2)$, $|\nu(w)(z)|^2 \to 1$, $e(w; \sigma_0, \rho) \to \frac{1}{2}l_\rho(\gamma)^2$ *and* $\mathscr{J}(w; \sigma_0, \rho) \to 0$ *in* $C^{1,\alpha}$ *as* $y \to \infty$ $(z \to p)$. *Furthermore, in the coordinates defined above for* $M \sim p \sim M_1^c$ *and* $U \subset (N, \rho|dw|^2)$, $\nu(w)(z) \to 1$ *in* $C^{1,\alpha}$, *as* $y \to \infty$.

Remark. The above shows that the harmonic map $w: (M, \sigma_0|dz|^2; p) \to (N \sim \gamma, \rho|dw|^2)$ is actually the solution to a "Dirichlet problem" or "pair of Dirichlet problems" (depending on whether γ does not separate or separates N, respectively), where by the "Dirichlet problem" we mean the problem of finding a harmonic $w: M \sim p \to N \sim \gamma$, so that, in the natural coordinates, $\lim_{z\to\infty, y\in\partial_i M_1^c} w(x, y) = \varphi_i(x)$, where φ_i are two appropriate constant speed parametrizations of γ.

Now there is an action of $\mathbf{R}$ on the Teichmüller space T_g corresponding to cutting N along Γ and reidentifying the two boundary components of $N \sim \gamma$ by a constant speed map $\psi^\theta: \gamma \to \gamma$; clearly, there is a family of such identifications, N_θ, parametrized by the distance to the left between

p and $\varphi(p)$. There is a natural isometric map $j_\theta\colon (N \sim \gamma, \rho) \to (N^\theta \sim \gamma, \rho^\theta)$ which does not extend continuously to γ unless $\theta = 0$. (Because of the homogeneity of the constant curvature metric in a neighborhood of γ, the constant -1 curvature metric ρ on $N \sim \gamma$ extends to a new constant curvature metric ρ^θ on the reidentified N^θ.)

We see that $(N^\theta \sim \gamma, \rho^\theta)$ and $(N \sim \gamma, \rho)$ are isometric, and (N^θ, ρ^θ) and (N, ρ) differ only in the identification of the boundary components of $N \sim \gamma$. Thus since $w\colon (M, \sigma_0|dz|^2, p) \to (N, \rho)$ actually has image in $N \sim \gamma$, we see that we can create a new harmonic map $w_\theta\colon (M, \sigma_0|dz|^2; p) \to (N^\theta, \rho^\theta)$ by setting $w_\theta = j_\theta \circ w$.

Recall that a neighborhood U of the node p in $(M, \sigma|dz|^2; p)$ is complex isomorphic to $\{(z, w) \in \mathbf{C}^2 | zw = 0,\ z(p) = 0 = w(p)\}$. In particular, $U \sim p$ has two components, U_1 and U_2, so that on U_1 we have a local uniformizing coordinate z with $z(p) = 0$. Let Φ be a meromorphic quadratic differential on U, with a second order pole at p. The differential Φ then admits the expansion in coordinates

$$\Phi = \Phi(z)\,dz^2 = (a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + \cdots)\,dz^2$$

on U_1; in this case, we note that the coefficient a_{-2} is independent of coordinates, and so it makes sense to speak of a_{-2} as the leading coefficient of Φ at p.

Proposition 3.7. *For the harmonic diffeomorphism $w\colon (M, \sigma_0|dz|^2; p) \to (N, \rho|dw|^2)$, the quadratic differential $(w^*\rho)^{2,0}$ is holomorphic on $M \sim p$ and meromorphic on $(M, \sigma|dz|^2; p)$ with a second order pole at the node p. The leading coefficient of $(w^*\rho)^{2,0}$ at p is positive and real, and equals $\frac{1}{4}l_\rho(\gamma)^2 = (w^*\rho)^{2,0}$.*

Proof. By Proposition 3.4, the Jacobian $\mathscr{J}(w) > 0$ on $M \sim p$. Then, by our discussion following equation (2.1), $\Phi(z)\,dz^2 = (w^*\rho)^{2,0}$ is holomorphic on $M \sim p$. To investigate the behavior of $\Phi(z)\,dz^2$ near the node p, we recall the Bochner-type equation (2.2a),

$$\Delta_{\sigma_0} \log \mathscr{H} = 2\mathscr{J} + 2K(\sigma_0). \tag{3.9}$$

Suppose first that γ separates N into two surfaces with genera g_1 and g_2, where $g_1 + g_2 = g$; then p separates M into two surfaces of genus g_1 and g_2. We first integrate the left-hand side of (3.9) over the domains

M_s^c and then let $s \to \infty$. First, we find that

$$
\begin{aligned}
&\iint_{M_s^c} \Delta_{\sigma_0} \log \mathscr{H}\, dA(\sigma_0)\\
&= \iint_{M_s^c} \Delta_{\sigma_0} \log(\mathscr{H}\sigma_0)\, dA(\sigma_0) - \iint_{M_s^c} \Delta_{\sigma_0} \log \sigma_0\, dA(\sigma_0)\\
&= \int_{\partial M_s^c} \nabla \log(\mathscr{H}\sigma_0)\cdot n\, ds(\sigma_0) + 2\iint_{M_s^c} K(\sigma_0)\, dA(\sigma_0)\\
&\qquad\qquad (\text{since } 2K(\sigma_0) = -\Delta_{\sigma_0}\log\sigma_0)\\
&= \int_{\partial M_s^c} \nabla \log \Phi \cdot n\, ds(\sigma_0) - \int_{\partial M_s^c} \nabla \log \overline{\nu}\cdot n\, ds(\sigma_0)\\
&\quad + \iint_{M_s^c} K(\sigma_0)\, dA(\sigma_0) \quad (\text{using } \Phi(z) = \sigma_0(z)\mathscr{H}(z)\overline{\nu}(z))\\
&= 2\pi Z_s - \int_{\partial M_s^c} \nabla \log\overline{\nu}\cdot n\, ds(\sigma_0) + 2\iint_{M_s^c} K(\sigma_0)\, dA(\sigma_0),
\end{aligned}
\tag{3.10}
$$

where Z_s represents the number of zeros of Φ in M_s^c, counted with multiplicity. Lemma 3.6 shows that the second term of (3.10) vanishes as $s \to \infty$. On the other hand, using that $d_N(f(z), w(z)) \to 0$ for the harmonic map $f\colon (M \sim M_r^c, \sigma_0|dz|^2) \to (N, \rho|dw|^2)$ of a neighborhood of the nodes into a neighborhood of γ, we see that when we integrate the first term on the right-hand side of (3.9) we get

$$
\begin{aligned}
2\iint_{M_s^c} \mathscr{J}(w)\, dA(\sigma_0) &= 2A_\rho(N) + o(s)\\
&= 4\pi(2g-2) + o(s)\\
&= 2\pi((4g_1 - 4) + (4g_2 - 4) + 4) + o(s)\\
&= 2\pi(Z_1 - P_1 + Z_2 - P_2 + 4) + o(s),
\end{aligned}
\tag{3.11}
$$

where Z_i and P_i represent the number of zeros and poles, counted with multiplicity, of a meromorphic quadratic differential on a compact surface of genus g_i. We next note that for s sufficiently large, Φ has all its zeros in M_s^c. This follows because $\Phi = \sigma_0(z)\mathscr{H}(z)\overline{\nu}(z)$ and, in the natural coordinates, $\sigma_0(z) = 1$ near p while, by the proof of Proposition 3.7, $\mathscr{H}(z) \to \frac{1}{4}l_\rho(\gamma)^2$ and $|\nu(z)| \to 1$ as $y \to \infty$. Therefore for s sufficiently large, $Z_1 + Z_2 = Z_s$. So, if we choose s sufficiently large, and compare (3.9), (3.10), and (3.11), we conclude that $P_1 + P_2 = 4$.

Consider the map w restricted to a neighborhood

$$\{|z| < \varepsilon, |\zeta| < \varepsilon | z\zeta = 0\} \subset \mathbf{C}^2$$

near the node; we notice that w and $w \circ I$, where I is the involution $I: z \leftrightarrow \zeta$, and C^1 close, up to a rotation. Thus $\Phi(z) = \rho(w(z))w_z\overline{w_{\bar z}}$ must have pole orders $P_1 = P_2$, and therefore has second order poles at p.

We see that the leading coefficient of $(w^*\rho)^{2,0}$ is real because the leading coefficient of $(f^*\rho)^{2,0}$ is real, where f is the cylinder map, and $\|w - f\|_{C^1} \to 0$ as $y \to \infty$ $(z \to p)$.

To compute the leading coefficient of $(w^*\rho)^{2,0}$, we use that on the flattened cylinders $((M \sim p) \sim M_s^c, \sigma_0|dz|^2)$, the leading coefficient is

$$a_{-2} = \lim_{y\to\infty} \Phi(y) = \lim_{y\to\infty} \sigma_0 \mathscr{H}\overline{\nu}.$$

But since $\nu \to 1$ (with respect to the usual coordinates) and $\mathscr{H} + \mathscr{H}/|\nu|^2 = e \to l_\rho(\gamma)^2/2$ by Lemma 3.6, we find that $a_{-2} = \frac{1}{4}l_\rho(\gamma)^2$.

The quadratic differentials $(w^*\rho)^{2,0}$ and $(w^{\theta *}\rho^\theta)^{2,0}$ agree because on $M \sim p$,

$$w^{\theta *}\rho^\theta = (j^\theta \circ w)^* j_*^\theta \rho) = w^* j^{\theta *} j_{\theta *}\rho = w^*\rho;$$

equivalently, on the complement of the core geodesic, $(N \sim \gamma, \rho)$ and $(N^\theta \sim \gamma, \rho^\theta)$ differ only by an isometry.

The proof for γ not dividing N is similar.

Remark. The essence of the above proof is that w and the cylinder map f are C^1 close near p, and that the holomorphic quadratic differential for f has second order poles at p. However, we find the use of (2.2a) as a sort of index formula more enlightening.

We derive one final property of the harmonic map w. We prove

Proposition 3.8. *The harmonic map* $w: (M, \sigma_0|dz|^2; p) \to (N, \rho|dw|^2)$ *is asymptotic to the map* $f: (M \sim p \sim M_1^c, \sigma_0|dz|^2) \to (N, \rho|dw|^2)$ *with appropriate boundary values in the sense that* $d_N(f(z), w(z)) = O(e^{-c(N)y})$ *as* $y \to \infty$.

Remark. We recall that $y = \exp d_\sigma((x, y), (x, 1))$ so that, as maps between hyperbolic surfaces, w and f are rapidly asymptotic.

Proof. We prove the proposition in one component $\mathscr{C} = [0, 1] \times [1, \infty)$ of $M \sim p \sim M_1^c$. Let $\varphi: [0, 1] \to \gamma$ denote the limit in the obvious coordinates of the map $w(\partial M_s^c \cap \mathscr{C})$, and let $\beta(\varphi): (\mathscr{C}, \sigma_0|dz|^2) \to (N, \rho|dw|^2)$ denote the map described in coordinates on $\mathscr{C}$ as $\beta(\varphi)(x, y) = \varphi(x)$.

With f defined by (3.1) with boundary values φ, it is easy to compute that $d_N(f(z), \beta(\varphi)(z)) = O(e^{-2l/y})$, where $l = l_\rho(\gamma)$. We want to show that $d_N(w(z), \beta(z)) = O(e^{-c(N)y})$.

Recall the definition of $\nu = \nu(w)(z)$ as $\nu = w_{\bar{z}}/w_z$ with respect to the coordinates z on M and w on N. We know from Lemma 3.6 that in our coordinates, $\nu \to 1$ as $y \to \infty$. Our plan is to estimate the rate at which ν approaches its limit.

Recall the holomorphic quadratic differential $\Phi = (w^*\rho)^{2,0}$ of Proposition 3.7. We rewrite (2.2a) as

$$\begin{aligned}\Delta_{\sigma_0} \log(1/|\nu|) &= 2(|\Phi|/\sigma_0) \sinh \log(1/|\nu|) \\ &\geq (l_\rho(\gamma)^2/4) \log(1/|\nu|),\end{aligned} \tag{3.12}$$

since $|\Phi|/\sigma_0 \to l_\rho(\gamma)^2/4$ as $y \to \infty$. Now, by Lemma 3.6, $\log(1/|\nu|) \to 0$ as $y \to \infty$, so we can find an s so that $\log(1/|\nu|) \leq K$ on $M \sim p \sim M_s^c$. Thus on $M \sim p \sim M - s^c$, we may apply the maximum principle to (3.12) to conclude that $\log(1/|\nu|) \leq k_1 e^{-(l/2)y}$. Therefore $1 - |\nu| \leq k_2 e^{-(l/2)y}$ for $y \in \mathscr{C} \cap M \sim p \sim M_s^c$.

We next want to control the argument of $1 - \nu$. For this we notice that $\mathscr{C}$ is conformally a punctured disk, on which Φ is holomorphic with a second order pole at the puncture (node) p, with leading coefficient $\frac{1}{4} l_\rho(\gamma)^2$. Thus, working in the coordinates $\mathscr{C} \cong [0, 1] \times [1, \infty)$, we express $\Phi(z)\, dz^2$ as $\Phi(z) = (l_\rho(\gamma)^2/4 + O(e^{-2\pi y}))\, dz^2$. Now $\Phi = \sigma_0 \mathscr{H} \bar{\nu}$, so we conclude that relative to the coordinates we have chosen on $\mathscr{C}$, $\operatorname{Im} \nu = O(e^{-2\pi y})$ and

$$1 - \nu = O(e^{-(l/2)y}) \tag{3.13}$$

for $l < 4\pi$ since $1 - |\nu|^2 = O(e^{-(l/2)Y})$.

Next we compute that $|1 - \nu| = |(w_z - w_{\bar{z}})/w_z| = |w_y|/|w_z|$ and since, in our coordinates for $\mathscr{C}$, $w_z \to 1$ as $y \to \infty$, we conclude that $|w_y| = O(e^{-(l/2)y})$. So pick $z_0 = (x_0, y_0) \in \mathscr{C} \sim M_s^c$, and let Γ denote the path $\Gamma = \{(x_0, y) | y_0 \leq y < \infty\}$ in C; by our construction of $\beta(\varphi)$, we have that $\lim_{y \to \infty, z \in \Gamma} w(z) = \beta(\varphi)(z_0)$. Therefore

$$\begin{aligned} d_N(w(z), \beta(z)) &\leq \int_\Gamma \left\| w_* \frac{\partial}{\partial y} \right\|_\rho dy \leq \int_{y_0}^\infty c' \left| \frac{\partial w}{\partial y} \right| dy \\ &\leq c \int_{y_0}^\infty e^{-(l/2)y}\, dy = c e^{-(l/2)y_0}, \end{aligned}$$

the second inequality coming from our choice of coordinates for $U \supset \gamma$.

We conclude that $d_N(w(z), \beta(z)) = O(e^{-c(N)y})$ and thus that

$$d_N(w(z), f(z)) = O(e^{-c(N)y}). \quad \text{q.e.d.}$$

We now discuss the uniqueness of $w: (M, \sigma_0|dz|^2; p) \to (N, \rho|dw|^2)$. We prove

Proposition 3.9. *If u and w are homotopic harmonic diffeomorphisms on $M \sim p$ from $(M, \sigma_0|dz|^2; p)$ to $(N, \rho|dw|^2)$ whose quadratic differentials have second order poles at the nodes with positive real leading coefficients, then $u = w$.*

Proof. We begin by showing that u and w share important properties in the neighborhood of p; for convenience we work in only one component $\mathscr{C}$ of a deleted neighborhood of p in $M \sim p$. Now, for s sufficiently large, $\Phi(u) = a_{-2}(u) + O(e^{-2\pi y})$ there in the standard coordinates. So, in the obvious notation,

$$\begin{aligned}
\iint_{C\cap M_{s+1}\sim M_s} e(u)\, dA(\sigma_0) &= \iint_{\mathscr{C}\cap M_{s+1}\sim M_s} \mathscr{J}(u)\, dA(\sigma_0) \\
&\quad + 2\iint_{\mathscr{C}\cap M_{s+1}\sim M_s} \mathscr{L}(u)\, dA(\sigma_0) \\
&< \operatorname{Area}(N) + 2\iint_{\mathscr{C}\cap M_{s+1}\sim M_s} \mathscr{L}(w)\, dA(\sigma_0) \\
&\qquad\qquad \text{(because } u \text{ is a diffeomorphism)} \\
&< \operatorname{Area}(N) + 2\iint_{\mathscr{C}\cap M_{s+1}\sim M_s} \mathscr{L}(u)/|\nu(u)|\, dA(\sigma_0) \\
&\qquad\qquad \text{(because } |\nu(u)| < 1) \\
&= A(N) + 2\iint_{\mathscr{C}\cap M_{s+1}\sim M_s} |\Phi(u)|\, dx\, dy \\
&\qquad\qquad \text{(by equality of integrands)} \\
&< K \quad \text{(depending only on } a_{-2} \text{ but not on } s).
\end{aligned}$$

Again using the argument following (3.0), we find that $e(u) < k_1$ on $\mathscr{C} \sim M_s^c$. Since we also have $|\Phi(u)|/\sigma_0 < \mathscr{H}(u)$, we conclude that $0 < c_1^2 < \mathscr{H}(u) < k_1$ on $\mathscr{C} \sim M_s^c$. Consequently, $|\nu(u)| = |\Phi(u)|/(\sigma_0\mathscr{H}(u))$ is bounded from above and away from zero in $\mathscr{C} \sim M_s^c$. We repeat our argument of Proposition 3.8, first rewriting (2.2a) so that $|\nu(u)|$ satisfies

$$\Delta_{\sigma_0} \log 1/|\nu(u)| = 2(|\Phi(u)|/\sigma_0) \sinh \log 1/|\nu(u)| \geq 2c_1^2 \log 1/|\nu(u)|.$$

The maximum principle again shows that $\log 1/|\nu(u)| \leq ke^{-c_1 y}$ on $\mathscr{C} \sim M_s^c$ in our usual coordinates and so $(1 - |\nu(u)|) \leq k_2 e^{-c_1 y}$ on $\mathscr{C} \sim M_s^c$.

We use this to compute the ρ-lengths of the images under u of some curves in $\mathscr{C}\cap M_s^c$. Choose $z_0 = x_0 + iy_0 \in \mathscr{C}\cap M_s^c \cong [0, 1]\times[s, \infty)$. Then, since $[0, 1]\times\{y_0\}$ is a closed curve, $l_\rho(u([0, 1]\times\{y_o\})) < k_0$. Moreover, if $\Gamma_0 = \{x\}\times[y_0, \infty)$, we may use that $u^*\rho = 2\,\mathrm{Re}(\Phi(u)\,dz^2) + \sigma e(u)\,dz\,d\bar{z}$ to see that

$$\begin{aligned} l_\rho(u(\Gamma_0)) &= \int_{\Gamma_0} \left\| u_* \frac{\partial}{\partial y} \right\|_\rho ds_{\sigma_0} \\ &= \int_{\Gamma_0} \mathscr{H}^{1/2} - \mathscr{L}^{1/2} + O(\mathrm{Im}(\Phi(u))^2)\, ds_{\sigma_0} \\ &\le \int_{\Gamma_0} \mathscr{H}^{1/2}(1 - |\nu(u)|) + O(\mathrm{Im}(\Phi(u))^2)\, ds_{\sigma_0} \\ &\le k_2 \sup_{\Gamma_0} \mathscr{H}^{1/2} \int_{y_0}^\infty e^{-c_1 y}\, dy + k_2 \int_{y_0}^\infty e^{-4\pi y}\, dy \\ &\le k_4 \quad (\text{since } \mathscr{H}(u) < k_1), \end{aligned}$$

where k_4 depends only on z_0. We conclude that any two points in $\mathscr{C} \cap M_s^c$ may be connected by a curve Γ with $\mathscr{L}_\rho(u(\Gamma)) < 2(k_0 + k_4)$.

Now lift u to $\tilde{u}\colon (M \sim p)^\sim \to \widetilde{U}$, and suppose that $\tilde{z}$ and $\tilde{z}_0 \in M^*$, a particular representative of $M \sim p$ in $(M \sim p)^\sim$. Then we argue as we did for Lemma 3.6 that since $\tilde{u}(\tilde{z})$ is only a bounded distance from $\tilde{u}(\tilde{z}_0)$, and $\tilde{w}(\tilde{z})$ is only a bounded distance away from $\tilde{w}(\tilde{z}_0)$, we have $d_{\widetilde{U}}(\tilde{u}(\tilde{x}, \tilde{y}), \tilde{w}(\tilde{x}, \tilde{y})) < K$. Thus we can argue we did as for Proposition 3.8 to find that $\tilde{u}$ is rapidly asymptotic to a map $\tilde{f}$ of the form (3.1) for some choice of parametrization $\varphi(u)\colon S^1 \to \gamma$; in particular, $d_{\widetilde{U}}(\tilde{u}, \tilde{w}) \to$ const in C^1 as $\tilde{y} \to \infty$ and $a_{-2} = l_\rho(\gamma)^2/4$.

Finally, we build and project $(\tilde{u}, \tilde{w})$ as before to form the function

$$g\colon M \sim p \to (N \sim \gamma)^\sim \times (N \sim \gamma)^\sim / \pi_1 U$$

and the function $\overline{Q} = \cosh d_{(N\sim\gamma)^\sim \times (N\sim\gamma)^\sim} \circ g - 1$. Since we have $\overline{Q} \to$ const in C^1 as $y \to \infty$, $\|\nabla \overline{Q}\|_{\sigma_0} \to 0$ as $y \to \infty$ and so

$$\begin{aligned} 0 &\le \iint_{M_s^c} \overline{Q} \Delta_{\sigma_0} \overline{Q}\, dA(\sigma_0) \\ &= \iint_{M_s^c} -\|\nabla \overline{Q}\|_{\sigma_0}^2\, dA(\sigma_0) + \int_{\partial M_s^c} \overline{Q} \frac{\partial}{\partial n} \overline{Q}\, ds(\sigma_0) \\ &\le \int_{\partial M_s^c} \overline{Q} \|\nabla \overline{Q}\|_{\sigma_0}\, ds(\sigma_0) w \to 0 \end{aligned}$$

as $s \to \infty$. We conclude that $\overline{Q}$ is a constant, and so

$$d_{(N\sim\gamma)^\sim \times (N\sim\gamma)^\sim / \pi_1 (N\sim\gamma)^\sim}(\tilde{u}, \tilde{w})$$

is a constant. Thus the left-hand side of inequality (3.6) must vanish. But if $\overline{Q}$ were not identically zero, we would require $\mathrm{Nor}\, \tilde{w}_* e_1 = \mathrm{Nor}\, \tilde{w}_* e_2 = 0$, so that both $\tilde{w}_* e_1$ and $\tilde{w}_* e_2$ lay along the geodesic from $\tilde{w}(z)$ to

$\tilde{u}(z)$, and w could not be a diffeomorphism. So we see that $Q \equiv 0$ on M and $u = w$.

Remark. Our arguments can be easily generalized to create harmonic diffeomorphisms from $(M, \sigma|dz|^2; p)$ to $(N, \rho|dw|^2)$ defined on $M \sim p$ whose quadratic differentials have nonreal leading coefficients. We consider limits of harmonic maps $w_s^{\#}: (M_s, \sigma_s|dz|^2) \to (N, \rho|dw|^2)$ where $w_s^{\#}$ is in the homotopy class of maps obtained by first acting on M_s by $G(s) = \alpha s$ fractional Dehn twists and then composing with the original map w_s. Energy estimates which are only slightly more complicated than those following Lemma 3.2 show that $w_s^{\#}$ converges on M_r. The resulting limiting map $w^{\#}: (M, \sigma|dz|^2; p) \to (N, \rho|dw|^2)$ will have $\Phi^{\#} = ((w^{\#*}\rho)^{2,0})$ meromorphic with a second order pole at p. The leading coefficient of $\Phi^{\#}$ will have argument α and $w^{\#}$ will be asymptotic to an appropriately twisted explicit solution similar to (3.1). As in Proposition 3.9, such a map will be determined by the leading coefficient of its quadratic differential.

Alternatively, define a metric g by

$$g = w^*(\rho|dw|^2) = \Phi_0\, dz^2 + \sigma e(w)\, dz\, d\bar{z} + \overline{\Phi}_0\, d\bar{z}^2,$$

and then set

$$g_\alpha = e^{2i\alpha}\Phi_0\, dz^2 + \sigma e(w)\, dz\, d\bar{z} + e^{-2i\alpha}\overline{\Phi}_0\, dz^2$$

for $0 < \alpha < \pi/2$. Thus g_α defines a hyperbolic metric on a surface with geodesic boundary of length $l_\rho(\gamma)\cos\alpha$, where γ is the core geodesic on $(N, \rho|dw|^2))$. Furthermore the identity map $\mathrm{id}: (M, \sigma|dz|^2; p) \to (M, g_\alpha)$ is harmonic ($\mathscr{J}(\mathrm{id}) > 0$ and $(\mathrm{id}^* g_\alpha)^{2,0}$ is holomorphic) and the leading coefficient of $(\mathrm{id}^* g_\alpha)^{2,0} = e^{2i\alpha} l_\rho(\gamma)^2/4$. By applying a similar procedure to the harmonic map from $(M, \sigma|dz|^2; p)$ to an appropriately chosen hyperbolic metric with geodesic core length $l_\rho(\gamma)\sec\alpha$, it is possible to find the unique harmonic map $w_\alpha: (M, \sigma|dz|^2, p) \to (N, \rho|dw|^2)$ so that the second order pole of $(w_\alpha^*\rho)^{2,0}$ has leading coefficient with argument α.

Recall the maps $w_\theta: (M, \sigma_0|dz|^2; p) \to (N^\theta, \rho^\theta)$ described after Lemma 3.6.

Corollary 3.10. *The map $w_\theta: (M, \sigma_0|dz|^2; p) \to (N^\theta, \rho^\theta)$ is the unique harmonic diffeomorphism with positive real leading coefficients for its second order pole in its homotopy class.*

Remark 1. Recall the 'twisted' surfaces and metrics (N^θ, ρ^θ) described in the remarks following Lemma 3.6; consider a metric $\hat{\rho}^\theta$ on N (not N^θ) so that $(N, \hat{\rho}_\theta)$ represents the same point in T_g as (N^θ, ρ^θ). Then, as points in T_g, $\lim_{\theta\to l_p(\gamma)}(N, \hat{\rho}^\theta)$ and (N, ρ_0) are distinct. Consequently, if R were compact Riemann surface of genus g and $u_\theta: R \to (N, \hat{\rho}^\theta)$ were the unique harmonic map in a homotopy class of maps, then $\lim_{\theta\to 2\pi} u_\theta \neq u_0$. The problem is that u_0 and $\lim_{\theta\to l_p(\gamma)} u_\theta$ represents different homotopy classes of maps from R to (N, ρ_0). On the other hand, unlike the $\{u_\theta\}$, for the maps $\{w_\theta\}$ referred to in Corollary 3.10, $\lim_{\theta\to l_p(\gamma)} w_\theta = w_0$. Here $\lim_{\theta\to l_p(\gamma)} w_\theta$ and w_0 are in the same homotopy class of maps, since the image of w_θ omits γ, and thus Proposition 3.9 applies. In conclusion, while both $\{w_\theta\}$ and $\{u_\theta\}$ can be thought of as mapping into (but not onto) N, we can attribute the difference in the results $\lim_{\theta\to l_p(\gamma)} u_\theta \neq u_0$ and $\lim_{\theta\to l_p(\gamma)} w_\theta = w_0$ to the difference in the homotopy types of the image: $\pi_1(N \sim \gamma) \neq \pi_1 N$.

Remark 2. In the remark after Proposition 3.9, we required $g(s) \to \infty$ as $s \to \infty$, so that $d_{\tilde{U}}(\tilde{w}_s^\#, \tilde{w}_s) \to \infty$, otherwise our arguments would show that both could be approximated by the same maps f_s and would then converge to the same map w. Now when we constructed the harmonic map $w: (M, \sigma_0|dz|^2; p) \to (N, \rho)$ prior to Proposition 3.1, we did so as the limits of harmonic maps $w_s: (M_s, \sigma_s|dz_s|^2) \to (N, \rho|dw|^2)$. Proposition 3.9 and the remark following it now show that if we had constructed $\hat{w}: (M, \sigma|dz|^2; p) \to (N, \rho|dw|^2)$ as the limit of maps $\hat{w}_s: (\widehat{M}_s, \hat{\sigma}_s|d\hat{z}_s|^2) \to (N, \rho)$ in the proper homotopy classes with $(\widehat{M}_s, \hat{\sigma}_s|d\hat{z}_s|^2)$ limiting on $(M, \hat{\sigma}|d\hat{z}|^2; p)$ in Teichmüller space, and $(M, \hat{\sigma}|d\hat{z}|^2; p)$ is a surface with nodes conformally equivalent to $(M, \sigma|dz|^2, p)$, then we must have had $\hat{w} = w$.

This concludes the proof of Theorem 3.11, which was stated in the Introduction. q.e.d.

Since we are interested in constructing a deformation theory near a noded surface $(M, \sigma|dz|^2; \rho)$, we need to understand the behavior of maps $w: (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho|dw|^2; p_1, \cdots, p_n)$, where the maps send each node p_i to itself. These maps can be considered as maps between cusped hyperbolic surfaces of finite total energy. In this regard we have the following result of Jochen Lohkamp, the uniqueness statement of which is contained in [20].

Theorem 3.12 [15]. *In each homotopy class of maps containing a diffeomorphism* $w: (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho|dw|^2; p_1, \cdots, p_n)$, *taking p_i to p_j, there exists a unique harmonic diffeomorphism of finite total energy.*

Remark. In this situation, we can have nonuniqueness if we drop the requirement that all the maps have finite total energy. Indeed, consider the problem of mapping $([0, 1) \times [1, \infty), |dz|^2/y^2)$ to itself, where we identify $\{0\} \times \{y\}$ and $\{1\} \times \{y\}$ to obtain a half-infinite cylinder, with the boundary conditions on the map w being $w(x, 1) = (x, 1)$ and $\lim_{y\to\infty} w(x, y) = \infty$. The harmonic map $w(x, y) = u(x) + iv(y) = x+iv(y)$ satisfies $v(y)v''(y)-v'(y)^2+1 = 0$ and has one solution $v(y) = y$ of finite energy and a one parameter family

$$v_c(y) = (1/\sqrt{c})\sinh[\sqrt{c}(y-1) + \sinh^{-1}\sqrt{c}]$$

of infinite energy solutions.

Proposition 3.13. *The quadratic differential $(w^*\rho)^{2,0}$ of the map of finite total energy $w: (M, \sigma|dz|^2; p_1, \dots, p_n) \to (M, \rho|dw|^2; p_1, \dots, p_n)$ is holomorphic with poles of at most first order at the punctures $\{p_j\}$. Even if the quadratic differential is regular at p_j, but not identically zero, then the holomorphic energy $\mathscr{H}(w; \sigma_0, \rho) = (\rho(w)/\sigma_0(z))|w_z|^2$ is still singular at p_j, where $\sigma_0|dz|^2$ is a nonsingular, nonvanishing Euclidean metric in a neighborhood of p. If the quadratic differential vanishes everywhere, then $\mathscr{H}(w; \sigma_0, \rho) = (\rho(w)\sigma(z))|w_z|^2 \equiv 1$ for the hyperbolic metric $\sigma|dz|^2$; otherwise, $\mathscr{H}(w; \sigma_0, \rho) = (\rho(w)/\sigma(z))|w_z|^2$ is bounded on $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ from above, bounded below by 1, and has nodal limit $\lim_{z\to p}(\rho(w)/\sigma(z))|w_z|^2 = 1$.*

Proof. In a neighborhood of a puncture p, we consider a neighborhood U of p uniformized by the disk $\{0 < |z| < 1\}$, and we let $\sigma_0(z) \equiv 1$ in U so that the domain metric is flat. Let $\Phi = (w^*\rho)^{2,0}$, and let $\mathscr{H}_0(z)$ be the holomorphic energy density $(\rho(w)/\sigma_0(z))|w_z|^2$. Now, since $\mathscr{J}_0(w) = \mathscr{H}_0 - |\Phi|^2/(\sigma_0^2\mathscr{H}_0) > 0$, (2.2a) shows that $\Delta\mathscr{H}_0 \geq 2K(\sigma_0) \equiv 0$ in U (this is the estimate of Eells-Sampson [7]). So

$$\mathscr{H}_0(q) \leq \frac{1}{A(B_R(q))}\int_{B_R(q)} \mathscr{H}_0\, dA(\sigma_0) \leq \frac{E(w)}{A(B_R(q))},$$

where $B_R(q)$ is a ball of σ_0-radius R around $q \in U$. Thus for q near p, R can be chosen to be $\frac{1}{2}|z(q)|$. Using that $|\Phi| \leq \sigma_0\mathscr{H}_0$, we find $|z(q)^2\Phi(q)| \leq (4/\pi)E(w)$, and therefore Φ has at worst a second order pole at p since w has finite total energy. On the other hand,

$$\int_U |\Phi(z)|/\sigma_0(z)\,dz\,d\bar{z} \le \int_U \mathscr{H}_0\sigma_0\,dz\,d\bar{z} \le E(w)$$

so Φ can have pole of at most first order at p.

Recall that if Φ does not vanish identically in U. then Φ has a trajectory structure on U with the horizontal trajectories (the curves $\Phi(z)\,dz^2 > 0$) corresponding to the directions of maximal stretch, where well-defined, of the differential dw. In particular, there is a horizontal arc γ of Φ in U which terminates at p, and we choose coordinates (x, y) on U so that $\gamma = \{(x, 0)|0 \le x \le \varepsilon\}$ while $\sigma_0 \equiv 1$. Then, using again that $|\Phi| \le \sigma_0\mathscr{H}_0$ we can compute the ρ-length of $w(\gamma)$ to be

$$\begin{aligned} l_\rho(w(\gamma)) &= \int_0^\varepsilon (\mathscr{H}_0^{1/2} + |\Phi|/\mathscr{H}_0^{1/2})\,dx \\ &\le 2\int_0^\varepsilon \mathscr{H}_0^{1/2}\,dx \le 2\varepsilon \sup_\gamma \mathscr{H}_0^{1/2}. \end{aligned}$$

However, since w maps $U \sim p$ to a neighborhood of a hyperbolic puncture, $l_\rho(w(\sigma))$ must be infinite, so that $\mathscr{H}_0$ cannot be regular at p.

If $\Phi \equiv 0$, then $w^*\rho = \sigma(\rho/\sigma)|w_z|^2 = \sigma\mathscr{H}|dz|^2$ so that w is a conformal diffeomorphism, hence an isometry of the finite value hyperbolic surfaces $(M, \sigma|dz|^2)$ and $(M, \rho|dw|^2)$. We conclude $\mathscr{H} \equiv 1$.

Finally, suppose that $\Phi \ne 0$. We wish to show, for the hyperbolic metric $\sigma|dz|^2$ on $(M; p_1, \cdots, p_n)$, that $\mathscr{H}(z) = (\rho(w(z))/\sigma(z))|w_z(z)|^2$ is bounded from above and below by 1, and that $\lim_{z\to p_i} \mathscr{H}(z) = 1$. First we notice that a proof of the existence of the map w could be accomplished in a similar way as the proof of Proposition 3.1: we could consider maps between compact hyperbolic manifolds $w_s\colon (M_s, \sigma_s|dz|^2) \to (M, \rho_s|dw_s|^2)$ so that $E(w_s; \sigma_s, \rho_s) < K$, letting $\sigma_s|dz|^2 \to \sigma|dz|^2$ and $\rho_s|dw_s|^2 \to \rho|dw|^2$ (using the result of Bers in [5] that the metrics converge uniformly on compacta) to show that the maps converged in $C^{k,\alpha}(M_r)$ for every r. Then we would have

$$\mathscr{H}_s = (\rho_s(w_s)/\sigma_s(z))|w_s z|^2 \to (\rho(w)/\sigma(z))|w_z|^2 = \mathscr{H}$$

in $C^{k-1,\alpha}(M_r)$ for every r, and we could prove that $\mathscr{H} \ge 1$ on $(M, \sigma|dz|^2; p_1, \cdots, p_n)$, by showing that $\mathscr{H}_s \ge 1$ on $(M_s, \sigma_s|dz|^2)$. The last, however, follows easily from the maximum principle on the compact M_s applied to (2.2b), $\Delta_{\sigma_s} \log\mathscr{H} = 2\mathscr{H} - 2|\Phi|^2/\sigma_s^2\mathscr{H}) - 2$, where we use that $K(\sigma_s) \equiv K(\rho_s) \equiv -1$ (see [22, Lemma 5.1]).

To get a bound in the other direction, we want to apply the Schwarz lemma for harmonic maps of Goldberg and Har'El [9]. (We would like to thank Jürgen Jost for suggesting this.) It states that $e(w) \le 2K^2 A/B$, where Ricci $(\sigma|dz|^2) \ge -A$, the Gauss curvature of $\rho|dw|^2$ is bounded away from zero by $-B$, and w is a K-quasiconformal harmonic map. Recalling the definition of the Beltrami differential $\nu = w_{\bar{z}}/w_z$, K-quasiconformality means that $\sup_M (1+|\nu|)/(1-|\nu|) \le K < \infty$.

Let U_i be a small neighborhood of a node p_i; then $(U_i \sim p_i, \sigma|dz|^2)$ can be isometrically embedded in two copies of $(0<|\zeta|<1, |\zeta|^{-2}(\log|\zeta|)^{-2} \times |d\zeta|^{-2})$. Consider one of these copies and compute that in U_i, because Φ has a pole of at most first order,

$$|\nu| = \frac{|\Phi|}{\sigma\mathscr{H}} \le \frac{|\Phi|}{\sigma} = O\left(\frac{1}{|\zeta|} \cdot |\zeta|^2 (\log|\zeta|)^2\right) = O(|\zeta| \log|\zeta|^2).$$

In particular, $|\nu| \to 0$ as $\zeta \to p_i$. Since w is a diffeomorphism, we may conclude that $\mathscr{H}(1-|\nu|^2) = \mathscr{J} > c$ on the compact complement of the neighborhoods U_i of the nodes, so that $|\nu|$ is bounded away from 1 outside of neighborhoods of nodes as well. Thus w is K-quasiconformal, and we may apply the Schwarz lemma of [9] to conclude that $\mathscr{H} \le 2K^2$.

(We remark in passing that the maps of Theorem 3.11 are not K-quasiconformal for any K since $|\nu| \to 1$ as $\zeta \to p$ for those maps.)

We want to show that $\mathscr{H}(\zeta) \to 1$ as $\zeta \to p_i$. We note first that

$$|\Phi|^2/(\sigma^2\mathscr{H}) \le O(|\zeta|^{-2}|\zeta|^4(\log|\zeta|)^4) = O(|\zeta|^2(\log|\zeta|)^4).$$

Thus we can rewrite (2.2b) in U_i as

$$\Delta_\sigma \log\mathscr{H} = 2(\mathscr{H}-1) - 2|\Phi|^2/(\sigma^2\mathscr{H}) \ge 2\log\mathscr{H} - C(|\zeta|^2(\log|\zeta|)^4). \tag{3.14}$$

The map $z = -i\log\zeta$ sends $(0 \le |z| < a, |\zeta|^{-2}(\log|\zeta|^{-2})|d\zeta|^2)$ isometrically onto $D = (\{0 \le x < 1, y > \log a^{-1}\}, y^{-2}|dz|^2)$ with the usual identification. Relative to the flat metric $|dz|^2$ on D, inequality (3.14) becomes

$$y^2(\partial_x^2 + \partial_y^2)\log\mathscr{H} \ge 2\log\mathscr{H} - Cy^4 e^{-4\pi y}$$

so that

$$(\partial_x^2 + \partial_y^2)\log\mathscr{H} \ge \frac{2}{y^2}\log\mathscr{H} - Cy^2 e^{-4\pi y}.$$

Then $\log\mathscr{H}$ is dominated on $[0, 1)\times[b, b+L]$ by a solution of $f(x, y) = (x, v(y))$, where $v(y)$ satisfies $v''(y) - 2y^{-2}v(y) = -Cy^2 e^{-4\pi y}$ with boundary conditions $v(a) = v(b+L) = \sup_M \log\mathscr{H} = \log 2K^2$. One thus easily computes the solution to this equation and observes that

$v(y) <$ (const depending on b, $K)y^{-1}$. We conclude that $\log \mathscr{H} \to 0$ as $z \to p_i$ so that $\mathscr{H} \to 1$ as $z \to p_i$.

4. Coordinates for a deformation neighborhood of a noded surface

Our aim is to generalize Theorem 2.1 to the case where $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ is a surface with nodes. Let $w(\rho): (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho|dw|^2; p_{j_i}, \cdots, p_{j_m})$ be the unique harmonic map, diffeomorphic on $M \sim \{p_1, \cdots, p_n\}$, which fixes p_{j_i}, be in the homotopy class of the identity map on $M \sim \{p_1, \cdots, p_m\}$, and have positive real leading coefficients at any second order poles of its quadratic differential; its existence, uniqueness, and properties are guaranteed by an easy combination of the proof of Theorem 3.11, Theorem 3.12, and Proposition 3.13. The quadratic differential $\Phi(\rho) = (w(\rho)^*)^{2,0}$ is holomorphic on $(M \sim \{p_1, \cdots, p_n\}, \sigma|dz|^2)$ with pole possible at the nodes $\{p_1, \cdots, p_n\}$. Our goal is to parametrize the neighborhood $\mathscr{N}$ by these quadratic differentials and some auxiliary parameters $\{\tau_j\} \in (S^1)^n$.

We begin with some preparations. We first construct the space $\widehat{\mathscr{P}_g}$ on which we shall work. Recall that if p_i is a node on M, then a neighborhood of p_i is complex isomorphic to $U_i = \{(z_i, w_i) \in \mathbf{C}^2 | z_i w_i = 0, z(p_i) = 0 = w(p_i), |z_i| < \varepsilon, |w_i| < \varepsilon\}$. We can create a closed Riemann surface of genus g, say $M_{\vec{t}}$, $\vec{t} = (t_1, \cdots, t_n)$ near $\vec{0}$, by the following process: we first remove the neighborhood $\{|z_i| < |t_i|, |w_i| < |t_i|\}$ from each U_i to form an open Riemann surface $M^*_{\vec{t}}$, and then identify q_0 and q in $M^*_{\vec{t}}$ if q_0 and q lie in the domain of z_i and w_i, respectively, with $z_i(q_0) w_i(q) = t_i$. Now consider the curves γ_i on $M_{\vec{t}}$ given in coordinates by $\gamma_i = \{|z_i| = |w_i| = \sqrt{|t_i|}\}$. (Note that, if on $M_{\vec{t}}$, we collapse the curves γ_i to points, we obtain a space topologically equivalent to $(M, \sigma|dz|^2; p_1, \cdots, p_n)$.) Let T_i denote a Dehn twist around γ_i, and let $\Gamma = \Gamma(p_1, \cdots, p_n)$ denote the group of homotopy classes of diffeomorphisms generated by the homotopy classes of $T_1, \cdots, T_N$. The group Γ is a subgroup of the mapping class group Γ_g of $M_{\vec{t}}$, and is Abelian because the curves γ_i are disjoint. Moreover Γ acts freely on the Teichmüller space T_g, and so the quotient space $\mathscr{P}_g = \mathscr{P}_g(p_1, \cdots, p_n)$ is a smooth $(3g-3)$ dimensional complex manifold.

The manifold $\mathscr{P}_g$ is not compact, and we next describe a family of Riemann surfaces which represent a family of points leaving all of the compact sets of $\mathscr{P}_g$. There are two such descriptions. First we may choose

$3g-3-n$ linearly independent Beltrami differentials $\nu_1, \cdots, \nu_{3g-3-n}$ on $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ supported on a compact subset of $M \sim \bigcup U_i$, and then a vector $(\vec{s}, \vec{t}) = (s_1, \cdots, s_{3g-g-n}, t_1, \cdots, t_n)$ near the origin of $\mathbf{C}^{3g-3}$. Next consider the Beltrami differential $\sum s_i \nu_i$ on $M_{\vec{t}}$, and let $M_{(\vec{s}, \vec{t})}$ denote the Riemann surface which is the quasiconformal deformation of $M_{\vec{t}}$ determined by the Beltrami differential $\sum s_i \nu_i$; $M_{(\vec{s}, \vec{t})}$ is smooth if $|t_i| \neq 0$ for every $i = 1, \cdots, n$. In this way the coordinates $\{(s_1, \cdots, s_{3g-3-n}, t_1, \cdots, t_n) | t_1 \neq 0\}$ parametrize an open set in $\mathscr{P}_g$: a family $M_{(\vec{s}, \vec{t})}$ leaves every compact set in $\mathscr{P}_g$ if some of the t_i tend to zero.

Alternatively, choose curves $[\gamma_{n+1}], \cdots, [\gamma_{3g-3}]$ on the smooth surface M_t so that $M_t \sim \{\gamma_1, \cdots, \gamma_n, \gamma_{n+1}, \gamma_{3g-3}\}$ is a collection of pairs of pants. Next supposing that $(M_t, \sigma_t|dz_t|^2)$ represents $[\sigma_t] \in T_g$, let $(l_1(\sigma_t) = l_{\sigma_t}([\gamma_1]), \ldots, l_{3g-3}(\sigma_t) = l_{\sigma_t}([\gamma_{3g-3}))$ and $(\theta_1(\sigma_t), \ldots, \theta_{3g-3}(\sigma_t))$ represent the Fenchel-Nielsen length and gluing coordinates, respectively, for $[\sigma_t]$ (see [1]) with respect to $\gamma_1, \ldots, \gamma_{3g-3}$. Note that the set $\{p_1, \ldots, p_n\} \cup \{\gamma_{n+1}, \cdots, \gamma_{3g-3}\}$ is a pair of pants decomposition for the noded surface $(M, \sigma|dz|^2; p_1, \cdots, p_n)$, some pants of which are degenerate (i.e., have cusps instead of a geodesic boundary of positive length), and let $l_1(\sigma) = 0, \cdots, l_n(\sigma) = 0, l_{n+1}(\sigma) = l_\sigma([\gamma_{n+1}]), \cdots, l_{3g-3}(\sigma) = l_\sigma([\gamma_{3g-3}])$ and $\theta_{n+1}(\sigma), \cdots, \sigma_{3g-3}(\sigma)$ represent the Fenchel-Nielsen length and gluing coordinates for the noded surface $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ with respect to $\{p_1, \cdots, p_n\} \cup \{\gamma_{n+1}, \cdots, \gamma_{3g-3}\}$; here $\theta_1(\sigma), \cdots, \theta_n(\sigma)$ are undefined. Bers [5] showed that if $(M, \sigma_t|dz_t|^2)$ were a family of smooth hyperbolic Riemann surfaces without nodes with Fenchel-Nielsen coordinates $(\vec{l}(\sigma_t), \vec{\theta}(\sigma_t))$ tending towards $(\vec{l}(\sigma_t), \vec{\theta}(\sigma_t))$, then the hyperbolic metrics $\alpha_t|dz_t|^2$ could be chosen to converge to $\sigma|dz|^2$, uniformly on compacta. So let $\mathscr{B}_g = \mathscr{B}_g(p_1, \cdots, p_n)$ denote the set of noded Riemann surfaces $(M, \sigma'|dz'|^2; p_{j_1}, \ldots, p_{j_m})$ with nodes at a subset $\{p_{j_1}, \ldots, p_{j_m}\}$ of $\{p_1, \cdots, p_n\}$ and equipped with hyperbolic metrics $\sigma'|dz'|^2$; we form the space $\widehat{\mathscr{P}_g} = \mathscr{P}_g \cup \mathscr{B}_g$ with a topology given by convergence of Fenchel-Nielsen coordinates. We could equivalently describe the topology on $\widehat{\mathscr{P}_g}$ by declaring that the correspondence $M_{(\vec{s}, \vec{t})} \to (\vec{s}, \vec{t}) \in \mathbf{C}^{3g-3}$, $(\vec{s}, \vec{t})$ near the origin, is a neighborhood chart centered at $(M, \sigma|dz|^2; p_1, \ldots, p_n)$

and that similar coordinates are neighborhood charts at other elements of $\mathscr{B}_g$ (see [6]).

The point is that Fenchel-Nielsen coordinates provide a nice description of a neighborhood $\widehat{\mathscr{N}}$ of our original noded Riemann surface $(M, \sigma|dz|^2; p_1, \cdots, p_n) \in \widehat{\mathscr{P}}$. In particular, we can describe such a neighborhood $\widehat{\mathscr{N}} \subset \widehat{\mathscr{P}}$ as consisting of those hyperbolic surfaces $(M, \rho|dw|^2; p_1, \cdots, p_{j_m})$ whose nodes are a subset of $\{p_1, \cdots, p_n\}$ and whose Fenchel-Nielsen coordinates satisfy (a) $l_i(\rho) < \varepsilon$, $i = 1, \cdots, n$; (b) $\theta_i(\rho)$ undefined for $1 \le i \le n$ and $i \in \{j_1, \cdots, j_m\}$; (c) $0 \le \theta_i(\rho) < 2\pi$ when $1 \le i \le n$ and $i \notin \{j_1, \cdots, j_m\}$; and (d) $|\theta_i(\rho) - \theta_i(\sigma)| < \varepsilon$ for $n+1 \le i \le 3g-3$. Here, because we work in the quotient space $\mathscr{P}_g = T_g/\Gamma$, we identify points whose twist angles at the curves $\gamma_1, \cdots, \gamma_n$ (when defined) differ by 2π.

In this way, $\widehat{\mathscr{N}} \cap \widehat{\mathscr{P}}_g$ is the product of n punctured disks, one for each node p_i, with $6g-6-2n$ intervals, one for each curve in $\{\gamma_{n+1}, \cdots, \gamma_{3g-3}\}$. Note that $\mathscr{M}_g = \mathscr{P}_g/(\Gamma_g/\Gamma)$ and that $\mathscr{P}_g$ is a branched cover of $\mathscr{M}_g$. Let $\pi\colon \mathscr{P}_g \to \mathscr{M}_g$ be the covering map, let $b \in \mathscr{B}_g$, let $\widehat{\mathscr{N}}$ be a neighborhood of b in $\widehat{\mathscr{P}}_g$, and let $N = \pi(\widehat{\mathscr{N}} \sim \mathscr{P}_g)$ be an open set in $\mathscr{M}_g$. Then $\widehat{\mathscr{N}}$ is a branched cover of the closure of $\mathscr{N}$ in $\overline{\mathscr{M}}_g$.

Let $\widehat{\mathscr{N}}$ be a neighborhood of $(M, \sigma|dz|^2; p_1, \cdots, p_n)$ in $\widehat{\mathscr{P}}_g$, and equip $\widehat{\mathscr{N}}$ with the Fenchel-Nielsen coordinates $(\vec{l}, \vec{\theta})$, $0 \le \theta_i < 2\pi$ when defined, as described above. Let $\tau_j = \exp i\theta_j$. For $(M, \rho|dw|^2; p_{j_1}, \cdots, p_{j_m})$ representing $[\rho] \in \widehat{\mathscr{N}}$, let $QD_{-2,n}(\sigma) = QD_{-2}(\sigma)$ denote the $(6g-6-n)$-dimensional real vector space of quadratic differentials which are holomorphic on $(M \sim \{p_1, \cdots, p_n\}, \sigma|dz|^2)$, have poles of at most second order at $p_1, \cdots, p_n$, and are such that the leading coefficients a^i_{-2} (of any second order pole at p_i) satisfy $a^i_{-2} \in \mathbf{R}_+$. Here we implicitly assume that the leading coefficient at a node of a meromorphic differential has the same value when evaluated from the perspective of either component of a punctured neighborhood of the node.

Let $Q_n(\sigma) = QD_{-2,n} \times (S^1)^n/\sim$ denote the product of the above vector space with n copies of S^1, labelled S^1_i (one for each node p_i), modulo an equivalence relation:

$$(\Phi, \tau_1, \cdots, \tau_n) \sim (\Phi, \zeta_1, \cdots, \zeta_n) \qquad \text{if for each } i, \text{ either } a^i_{-2} = 0 \text{ or } \tau_i = \zeta_i.$$

We define a map $\Phi\colon \widehat{\mathscr{N}} \to Q_n(\sigma)$ by $\Phi([\rho]) = ((w(\rho)^*\rho)^{2,0},\ \tau_1(\rho), \cdots, \tau_n(\rho))$.

Theorem 4.1. *Φ is a homeomorphism onto a neighborhood of* 0 *in* $Q_n(\sigma)$.

Proof. First we prove injectivity. Suppose $\Phi = \Phi_1 = \Phi_2 \in QD_{-2,n}(\sigma)$, and consider the functions $\mathscr{H}_1$ and $\mathscr{H}_2$, where $\mathscr{H}_i = (\rho_i(w_i)/\sigma(z))|(w_i)_z|^2$, and $w_i\colon (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho_i|dw_i|^2; p_{j_1}, \cdots, p_{j_m})$ is the harmonic map corresponding to Φ_i. Now, if Φ has a pole of second order at p_j, then

$$\frac{\mathscr{H}_1}{\mathscr{H}_2} = \frac{|\Phi_1|/(\sigma|\nu_1|)}{|\Phi_2|/(\sigma|\nu_2|)} = \frac{|\nu_2|}{|\nu_1|}$$

and since, by Lemma 3.6, $|\nu_i| \to 1$ as $z \to p_j$, we see that $\mathscr{H}_1/\mathscr{H}_2 \to 1$ as $z \to p_j$. If Φ has a pole of first order or is regular at p_j, then by Proposition 3.13, we have that $\mathscr{H}_i \to 1$ as $z \to p_j$, so that $\mathscr{H}_1/\mathscr{H}_2 \to 1$ as $z \to p_j$. By taking the difference of two copies of (2.2b), we obtain

$$\Delta \log \mathscr{H}_1/\mathscr{H}_2 - 2 = 2[1 + |\Phi|^2/(\sigma^2\mathscr{H}_1\mathscr{H}_2)](\mathscr{H}_1 - \mathscr{H}_2). \tag{4.1}$$

Now, if $\mathscr{H}_1/\mathscr{H}_2$ were not identically unity, there would be an interior maximum for $\mathscr{H}_1/\mathscr{H}_2$, which, after possibly reversing the roles of $\mathscr{H}_1$ and $\mathscr{H}_2$, we can take to be greater than one. Thus the left-hand side of (4.1) would be nonpositive and the right-hand side positive. So $\mathscr{H}_1 = \mathscr{H}_2$. Thus $w_2 \circ w_1^{-1}$ is an isometry between $w_i(M - \{p_1, \cdots, p_n\})$ since

$$w_i^*\rho_i = \Phi_i\, dz^2 + \sigma(\mathscr{H}_i + |\Phi_i|^2/(\sigma^2\mathscr{H}_i))\, dz\, d\bar{z} + \overline{\Phi}_i\, d\bar{z}^2.$$

If Φ has a first order pole at p_j, then $w_i(p_j) = p_j$, and $w_2 \circ w_1^{-1}$ extends to an isometry in a neighborhood of p_j. Since $w_2 \circ w_1^{-1}$ is an isometry off of $M \sim \{p_1, \cdots, p_n\}$, the Fenchel-Nielsen coordinates corresponding to $\gamma_{n+1}, \cdots, \gamma_{3g-3}$ agree. Now, if Φ has a second order pole at p_j, and $\Phi_1 = \Phi_2$, then the leading coefficients a^j_{-2} of Φ_1 and Φ_2 agree. Moreover, in the usual coordinates for a component of a neighborhood of $M \sim p_j$, $l_{\rho_i}(w_i([0, 1) \times \{y\})) \to 2\sqrt{a^j_{-2}}$ as $y \to \infty$ and $w_i([0, 1) \times \{y\})$ tends to a constant speed parametrization of the ρ_i-geodesic representative $\gamma_j(\rho_i)$ of $[\gamma_j]$. Thus, the Fenchel-Nielsen length coordinates for ρ_1 and ρ_2 corresponding to $[\gamma_j]$ agree, and the gluing angles agree because $\tau_j(\rho_1) = \tau_j(\rho_2)$. Hence ρ_1 and ρ_2 have the same Fenchel-Nielsen coordinates, and represent the same point in $\widehat{\mathscr{P}}_g$. (Indeed, because $w_i([0, 1] \times \{y\})$ asymptotically parametrizes $\gamma_j(p_i)$ by arclength, under an identification of $(M, \rho|dw|^2; p_{j_1}, \cdots, p_{j_m})$ across $\gamma_j(\rho_i)$ prescribed by $\tau_j(\rho_1) = \tau_j(\rho_2)$, the maps $w_2 \circ w_1^{-1}$ extend across the geodesic $\gamma_j(\rho_1)$.

We are left to show the continuity of Φ. Let K be a compact set of M, disjoint from neighborhoods of the nodes. Let ρ_k be a family of metrics representing $\{[\rho_k]\} \subset \widehat{\mathscr{N}}$ with $\rho_k \to \rho_0$; suppose ρ_0 represents $[\rho_0] \in \widehat{\mathscr{N}}$. On K, a subsequence of the metric tensors $\{\rho_k\}$ converges to ρ_0 uniformly (for $\rho_0 \in \overline{\mathscr{M}_g}$, this follows from [5]). Moreover, because maps of cusped surfaces have finite total energy, by the constructions of Theorem 3.11 for infinite energy maps, a corresponding subsequence of the harmonic maps $w_k: (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho_k|dw_k|^2)$ have total energy on K uniformly bounded, say $E(w_k)|_K < C_1$. Thus since K has bounded geometry, from the argument after (3.0) we get our new standard estimate $e(w_k)(p) < C_2$ for $p \in K$, and, as usual, convergence of a subsequence of w_k in $C^{2,\alpha}$, uniformly on K, to a limit harmonic map $w_\infty: (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho_0)$. By using a compact exhaustion of M and a diagonal argument we construct a harmonic map $w_\infty: (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho_0|dw|^2)$. Our uniqueness results then show that w_∞ must equal $w_0: (M, \sigma|dz|^2; p_1, \cdots, p_n) \to (M, \rho_0|dw_0|^2)$. In particular, convergence of w_k to w_0 in C^1 guarantees that the $\Phi_k = (w_k * \rho_k)^{2,0}$ converges to $\Phi_0 = (w_0 * \varphi_0)^{2,0}$.

Remark 1. One might be disappointed by Theorem 4.1, which states that $Q_n(\sigma)$ provides coordinates for $\widehat{\mathscr{N}}$. Certainly one might have hoped that the coordinates for $\widehat{\mathscr{N}}$ would be given by a space which did not require a seemingly ad hoc addition of $(S^1)^n$ and a subsequent identification. Yet we claim that Theorem 4.1 is, in a sense, optimal: from a coordinate description of a manifold, one should expect a description of the tangent and cotangent bundles, and here $Q_n(\sigma)$ gives rise to a bundle which is isomorphic to $T\widehat{\mathscr{P}}_g$. On the other hand, the space $QD_{\text{reg}}(\sigma)$ of holomorphic quadratic differentials with second order poles whose leading coefficients agree at the nodes, while perhaps appearing to be a better hope for coordinates for $\widehat{\mathscr{N}}$, gives rise naturally to a bundle over $\widehat{\mathscr{P}}_g$ that is topologically distinct from $T\widehat{\mathscr{P}}_g$ and $T^*\widehat{\mathscr{P}}_g$.

We sketch very imprecisely the reasons for this, beginning with the description of the cotangent bundle of $\overline{\mathscr{M}_g}$ due to Masur [16] and elaborated upon by Wolpert [25]. The following discussion is local in $\widehat{\mathscr{P}}_g$, and so descends to a neighborhood of an element of $\mathscr{D}_g \subset \overline{\mathscr{M}_g}$ as long as

$[(M, \sigma|dz|^2; p_1, \cdots, p_n)]$ is not an element of the branching locus of the covering $\pi: \widehat{\mathcal{N}} \to \pi(\widehat{\mathcal{N}}) \subset \overline{\mathcal{M}_g}$. Since the (complex) coordinates for $\widehat{\mathcal{N}}$ are $s_1, \cdots, s_{3g-3}, t_1, \cdots, t_n)$, a local description in $\widehat{\mathcal{N}}$ of the cotangent bundle of $\widehat{\mathcal{P}_g}$ is given by $ds_1, \cdots, ds_{3g-3-n}, dt_1, \cdots, dt_n$. Of course, one would prefer a more intrinsic description: the cotangent space to $\mathcal{P}_g$ at an unnoded surface (M, ρ) is given by $QD(\rho)$, the tangent space by equivalence classes of the space of Beltrami differentials $\mathrm{Belt}(\rho)$, and we would like such a description valid for $\widehat{\mathcal{N}}$. To this end (suppose for convenience that $n = 1$), Masur explicitly computes $\partial/\partial t|_{t\neq 0}$ as a Beltrami differential on an annulus $\{zw = t; |t| < |z| < 1, |t| < |w| < 1\}$ and finds that

$$\frac{\partial}{\partial t} = \frac{1}{2t}\frac{1}{\log|t|}\frac{z}{\overline{z}}\frac{d\overline{z}}{dz}.$$

The factor $(\log|t|)^{-1}$ is present because of our choice of coordinates for the annulus but the factor of t^{-1} involves the geometry of $T^*\widehat{\mathcal{P}_g}$. Now if we suppose that the cotangent covector $dt = f(t, z)\varphi$, where $\varphi = a(z)(dz/z)^2$ is a regular quadratic differential on the surface $M_{(\vec{s}, t)}$, then we can compute $f(t, z)$ by using the explicit pairing between Beltrami differentials and quadratic differentials: $(\nu, \varphi) = \int_M \nu\varphi$. This yields (as $|t| \to 0$)

$$\begin{aligned} 1 &= \left(dt, \frac{\partial}{\partial t}\right) \\ &= \int_{|t|<|z|<1} \frac{1}{2t}\frac{1}{\log|t|}\frac{z}{\overline{z}} f(t, z)\frac{a(z)}{z^2}\, dz\, d\overline{z} + O(1), \end{aligned}$$

and we find that we may suppose $f(t, z) = t + O(|z|)$ and $a(z) = 1 + O(|z|)$. Thus we can represent $dt = t\varphi$, where $\varphi \varepsilon QD_{\mathrm{reg}}(M_{(\vec{s}, t)})$. From this we see that QD_{reg} cannot represent $T^*\widehat{\mathcal{P}_g}$: on the overlap $\mathcal{P}_g \cap \widehat{\mathcal{N}}$, the identity transition function between $QD_{\mathrm{reg}}|_{\widehat{\mathcal{N}}}$ and $QD_{\mathrm{reg}}|_{\mathcal{P}_g}$ does not involve the factor t required by Masur's computation, and so the isomorphism between QD_{reg} and $T^*\widehat{\mathcal{P}_g}$ on $\mathcal{P}_g \cap \widehat{\mathcal{N}}$ cannot extend to an isomorphism on $\widehat{\mathcal{P}_g} \cap \widehat{\mathcal{N}}$.

One explanation for the factor t in Masur's formula $dt = t\varphi$, $\varphi \varepsilon QD_{\mathrm{reg}}(M_{(\vec{s}, t)})$, is similar to an explanation of why the harmonic map coordinates for $\widehat{\mathcal{N}}$ involve $(S^1)^n$: if we change coordinates for $\widehat{\mathcal{N}}$ for $(\vec{s}, t, \overline{t})$ to $(\vec{s}, |t|, \arg t)$ and use Masur's formula $dt = t\varphi$, we find that $d|t| = |t|\operatorname{Re}\varphi$ and $d(\arg t) = \operatorname{Im}\varphi$. Thus these differentials have expressions that are independent of the argument $\arg t$, in the same way that the harmonic maps to (N^θ, ρ^θ) are independent of θ.

Finally, we claim that $Q_n(\sigma)$ naturally gives rise to a bundle Q_n^T which is naturally dual to $T^*\widehat{\mathscr{P}_g}$ and so represents $T\widehat{\mathscr{P}_g}$. Suppose again that $n = 1$, and let $QD_{-2,n}(\sigma)$ have a real basis $\Phi_0, \psi_1, \cdots, \psi_{6g-8}$, where Φ_0 has a second order pole at the node p and $\operatorname{ord}_p \psi_i \geq -1$. Then in $\widehat{\mathscr{N}}$, we have the coordinates $(\Phi = l\Phi_0 + \sum \alpha_i \psi_i, \tau)$; we describe Q_n^T over $\widehat{\mathscr{N}}$ as the trivial bundle spanned over $\widehat{\mathscr{N}} \sim \mathscr{B}_g$ by $(\partial/\partial l, \partial/\partial \alpha_i, \partial/\partial \tau)$. Such a description easily patches together for neighborhoods of $\widehat{\mathscr{P}_g} \sim \mathscr{B}_g$, so we are left to consider a subneighborhood of $\widehat{N} \cap \mathscr{P}_g$ defined by $\{\varepsilon^2 < l < \varepsilon\}$, where we need to provide an intrinsic description of $Q_n^T|_{\widehat{\mathscr{N}}}$ as well as overlap transition functions from $Q_n^T|_{\widehat{\mathscr{N}}}$ to $T\mathscr{P}_g|_{\mathscr{P}_g}$. Now $T\mathscr{P}_g|_{\mathscr{P}_g}$ is given by formula (2.5) as $QD_{\mathrm{reg}}|_{\mathscr{P}_g}$, so we only need to identify $\partial/\partial l$, $\partial/\partial \alpha_i$, and $\partial/\partial \tau$ with quadratic differentials on $\rho \varepsilon \mathscr{P}_g$.

This we accomplish via standard Teichmüller theory: let $\nu(\rho(l, \vec{\alpha}, \tau))$ be the Beltrami differential for the harmonic map $w(\rho(l, \vec{\alpha}, .\tau))$: $(M, \sigma|dz|^2; p) \to (M, \rho(l, \vec{\alpha}, \tau))$. Then Ahlfors provides a natural way [2, formula (1.12)], to represent $(\partial/\partial l)(\nu(\rho(l, \vec{\alpha}, \tau)))$ and $(\partial/\partial \alpha_i)(\nu(\rho(l, \vec{\alpha}, g\tau)))$ as elements of $\operatorname{Belt}(\rho(l, \vec{\alpha}, \tau))$; we will denote these elements as $\nu_*(\partial/\partial l)$ and $\nu_*(\partial/\partial \alpha_i)$, respectively. Ahlfors provides another natural way to represent $\nu_*(\partial/\partial l)$ and $\nu_*(\partial/\partial \alpha_i)$ as elements of $QD(\rho(l, \vec{\alpha}, \tau))$; we denote those elements as $\Phi_*(\partial/\partial l)$ and $\Phi_*(\partial/\partial \alpha_i)$, respectively.

The important point about this process is that because the harmonic maps are independent of τ, the differentials $\nu_*(\partial/\partial l)$ and $\nu_*(\partial/\partial \alpha_i)$ will be independent of τ. It is not hard to represent $\partial/\partial \tau$ as a Beltrami differential $\nu_*(\partial/\partial \tau) \in \operatorname{Belt}(\rho(l, \vec{\alpha}, \tau))$ which is also nearly independent of τ; once again we may represent $\nu_*(\partial/\partial \tau)$ as a quadratic differential $\Phi_*(\partial/\partial \tau) \varepsilon QD(\rho(l, \vec{\alpha}, \tau))$.

We now show that Q_n^T is naturally dual to $T^*\widehat{\mathscr{P}_g}$. From our construction, it is not difficult to see that it suffices to find a nondegenerate pairing between Q_n^T and $T^*\widehat{\mathscr{P}_g}$ on $\{\varepsilon^2 < l < \varepsilon\}$ that extends to a nondegenerate pairing on $\widehat{\mathscr{N}}$. To this end, let $\psi \varepsilon T^*\widehat{\mathscr{P}_g}$ and $\Phi \varepsilon Q_n^T$ be holomorphic quadratic differentials on the same base surface $\rho \varepsilon \mathscr{P}_g$ and define $\langle \psi, \Phi \rangle_\rho = \int_M \psi \overline{\Phi}/\rho^2 \, dA(\rho)$.

Next, we need to relate the $(|t|, \arg t)$ coordinates that we used to define $T^*\widehat{\mathscr{P}_g}$ and the (l, θ) coordinates that we used to define Q_n^T.

In a future article, we will show that for $|t|$ small, the change of coordinate Jacobian $\partial(|t|, \arg t)/\partial(l, \theta)$ is nearly diagonal on $\widehat{\mathscr{N}} \cap \mathscr{P}_g$, i.e., that the off-diagonal terms are of lower order than the diagonal terms. We are now in a position to use our formulas $d|t| = |t| \operatorname{Re} \varphi$ and $d(\arg t) = \operatorname{Im} \varphi$, where φ was independent of t, and, of course, independent of τ.

So, finally, consider for example

$$\langle d|t|, \Phi_*(\partial/\partial l)\rangle_{\rho(l, \vec{\alpha}, \tau)} = \int_M |t| \operatorname{Re} \varphi \Phi_*(\partial/\partial l)/\rho^2 \, dA(\rho).$$

Each entry in the integrand is nearly independent of τ. Since a similar argument also applies to $\langle d|t|, \Phi_*(\partial/\partial \tau)\rangle$ and $\langle \cdot, \partial/\partial \alpha_i\rangle$, and by construction $\langle d|t|, \Phi_*(\partial/\partial \tau)\rangle$ and $\langle d(\arg t), \Phi_*(\partial/\partial l)\rangle$ are comparatively small, we conclude that the inner product $\langle \cdot, \cdot \rangle_\rho$ between Q_n^T and $T^*\widehat{\mathscr{P}}_g$ (while itself unbounded on $\mathscr{P}_g$) can be normalized to admit a nondegenerate extension to all of $\widehat{\mathscr{P}}_g$. Thus Q_n^T is dual to $T^*\widehat{\mathscr{P}}_g$.

Remark 2. The situation of Proposition 3.13 contrasts with that of Theorem 2.1. Let w be a harmonic map $w: (M, \sigma|dz|^2; p) \to (M, \rho|dw|^2)$. If $(w^*\rho)^{2,0}$ has a pole at p, then $(M, \rho|dw|^2)$ is a cusped surface or a surface with geodesic boundary of positive length. On the other hand, if $(w^*\rho)^{2,0}$ is regular at p, then, letting $\sigma_0|dz|^2$ be a nonsingular, nonzero metric in a neighborhood of p, $\mathscr{H}_0(z) = (\rho(w)/\sigma_0(z))|w_z|^2$ is either regular or singular.

If $(M, \rho|dw|^2)$ is compact and boundaryless, $\mathscr{H}_0(z)$ is regular at p (Theorem 2.1 and [22]), and if $(M, \rho|dw|^2)$ is cusped, $\mathscr{H}_0(z)$ is singular at p. This suggest a picture of the degeneration of a surface of genus g in terms of solutions to (2.2b) on a surface $(M, \sigma|dz|^2, p)$: holomorphic quadratic differentials with poles of order two correspond to points in $\mathscr{P}_g$, obtained by opening the node p; those with poles of order one to points of $\mathscr{B}_g$; those which are regular but whose solution $\mathscr{H}_0(z)$ to (2.2b) is singular with respect to $\sigma_0|dz|^2$ correspond to points on $\mathscr{B}_g$; and those which are regular but whose solution $\mathscr{H}_0(z)$ to (2.2b) is regular with respect to $\sigma_0|dz|^2$ correspond either to a point in $\mathscr{M}_{g-1}$ or to a pair of surfaces, one an element of $\mathscr{M}_{g_1}$, the other an element of $\mathscr{M}_{g-g_1}$.

5. Real analyticity of the hyperbolic metrics

We next consider the dependence of the hyperbolic metrics $\rho \in \widehat{\mathscr{N}}$ upon their coordinates $\Phi(\rho) \in Q_n(\sigma)$. In view of the representation (2.1), it is enough to understand the dependence of $\mathscr{H}(\Phi(\rho)) = \mathscr{H}(w(\rho); \sigma, \rho)$ upon the coordinates $\Phi(\rho) \in Q_n(\sigma)$. We note that we can unambiguously write $\mathscr{H}(\Phi(\rho))$ as $\mathscr{H}(\Phi)$ for $\Phi \in QD_{-2,n}(\sigma)$ since $\mathscr{H}(\Phi(\rho_1)) = \mathscr{H}(\Phi(\rho_2))$ as long as $(w(\rho_1)^*\rho_1)^{2,0} = (w(\rho_2)^*\rho_2)^{2,0}$, independent of the coordinates (τ_i) used to parametrize the identification of $w(\rho)(M \sim \{p_1, \cdots, p_n\})$ across the geodesic boundaries.

Choose $\Phi_0 \in QD_{-2,n}(\sigma)$ with $\Phi_0 \neq 0$.

Let $\mathscr{H}(t) = \mathscr{H}(t\Phi_0)$. Note that here the parameter t is real and positive; also it is different from the variable of Masur which we used in §4 as a complex variable for $\widehat{\mathscr{N}}$. In analogy with Theorem 2.2, we prove

Theorem 5.1. *$\mathscr{H}(t)$ is real analytic in $\sqrt{t}$.*

Proof. We argue as in the proof of Theorem 2.2, with some technical modifications necessary because of the noncompactness of $M \sim \{p_1, \cdots, p_n\}$, the vanishing of the injectivity radius of $(M \sim \{p_1, \cdots, p_n\}, \sigma|dz|^2)$ as $z \to p_i$, and the existence of a nontrivial kernel for $\Delta_\sigma - 2$. We restrict ourselves to the case where $(M, \sigma|dz|^2; p)$ has a cusp only at p, the general case following from a suitable infusion of notation.

As usual, our principal tool is the explicit solution (3.1) of the case when the domain $(P, \sigma|dz|^2)$ is a half-infinite cylinder, the target $(N, \rho|dw|^2)$ is a hyperbolic cylinder with geodesic boundary of hyperbolic length l, and f solves the harmonic mapping problem with appropriate boundary conditions. Here we will choose each component of $(P \sim p, \sigma|dz|^2)$ to be isometrically $P_a \cong ([0, 1) \times (a, \infty), y^{-2}|dz|^2)$, where $a > 1$. In that case, $(f^*\rho)^{2,0} = \frac{1}{4}l^2\,dz^2$ and the holomorphic energy

$$h(l) = \mathscr{H}(f; \sigma, \rho) = \frac{1}{4}\frac{l^2}{\sigma}\left[\frac{1 + e^{l(1-y)}\sqrt{(1-l)/(1+l)}}{1 - e^{l(1-y)}\sqrt{(1-l)/(1+l)}}\right]^2. \tag{5.1}$$

One sees that for fixed y, $h(l)$ is real analytic in l (for l small), and a straightforward computation confirms that, for fixed y, $h(l) \to 1$ as $l \to 0$. Moreover, $h(l)$ satisfies (2.2) on the cylinder, i.e.,

$$\Delta \log h(l) = 2h(l) - \frac{2(l^2/4)^2}{\sigma^2 h(l)} - 2. \tag{5.2}$$

With these facts in mind, we construct an approximation $\mathscr{K}(t)$ to $\mathscr{H}(t)$, using appropriate functions $h(l)$ in the cusps. Let a_{-2} denote the leading coefficient of the second order pole of Φ_0 at the node p, where we set $a_{-2} = 0$ if Φ_0 has a pole of first order or is regular at p. We consider a neighborhood U about the node p so that U is isometrically two copies of P_a for some a. We then define $\mathscr{K}(t)$ in U to be $h(2(a_{-2})^{1/2})$, where by $h(0)$ we mean the function which is identically unity. We then extend $\mathscr{K}(t)$ over the rest of M so that it is nonzero, C^∞ in z, and real analytic in $\sqrt{t}$; we also require that $\mathscr{K}(0) \equiv 1$. (One way to do this is to choose ε so that $P_{a-\varepsilon}$ isometrically embeds in $(M, \sigma|dz|^2)$, set $\mathscr{K}(t)$ to be identically one in $M \sim \bigcup\{\text{image}(P_{a-\varepsilon})\}$ and then extend $\mathscr{K}(t)$ smoothly across the annuli $\bigcup\{P_{a-\varepsilon} \sim P_a\}$.)

We consider the functions $g(t) = \mathscr{H}(t)/\mathscr{K}(t)$ and $\mathscr{G}(t) = \log g(t)$. The two subcases are when Φ_0 has a second order pole at p and when it does not.

Suppose that $a_{-2} \neq 0$. Then from (5.2), $\mathscr{K}(t)$ satisfies

$$\Delta \log \mathscr{K}(t) = 2\mathscr{K}(t) - 2t^2(a_{-2})^2/(\sigma^2\mathscr{K}(t)) - 2 + \alpha(t), \tag{5.3}$$

where $\alpha(t)$ is the error coming from extending $\mathscr{K}(t)$ over $M \sim U$; we see that $\alpha(t)$ is supported in the complement of U. From (2.2) and (5.3) it follows that

$$\begin{aligned}
\Delta\mathscr{G}(t) &= \Delta \log g(t) = \Delta \log \mathscr{H}(t) - \Delta \log \mathscr{K}(t) \\
&= 2(\mathscr{H}(t) - \mathscr{K}(t)) - \frac{2t^2(|\Phi_0|^2 - a_{-2}^2)}{\sigma^2\mathscr{H}(t)} \\
&\quad - \frac{2t^2a_{-2}^2}{\sigma^2\mathscr{H}(t)} + \frac{2t^2a_{-2}^2}{\sigma^2\mathscr{K}(t)} - \alpha(t) \\
&= 2\mathscr{K}(t)(e^{\mathscr{G}(t)} - 1) - \frac{2t^2a_{-2}^2}{\sigma^2\mathscr{K}(t)}(e^{-\mathscr{G}(t)} - 1) \\
&\quad - \frac{2t^2(|\Phi_0|^2 - a_{-2}^2)e^{-\mathscr{G}(t)}}{\sigma^2\mathscr{K}(t)} - \alpha(t) \\
&= 2\mathscr{K}(t)(e^{\mathscr{G}(t)} - 1)\left(1 + \frac{t^2a_{-2}^2e^{-\mathscr{G}(t)}}{\sigma^2\mathscr{K}(t)^2}\right) \\
&\quad - \frac{2t^2(|\Phi_0|^2 - a_{-2}^2)e^{-\mathscr{G}(t)}}{\sigma^2\mathscr{K}(t)} - \alpha(t).
\end{aligned} \tag{5.4}$$

In the proof of Theorem 2.2, we analyzed (2.2) for $\mathscr{H}(t)$; for the present theorem, we will analyze (5.4) for $\mathscr{G}(t)$. As in Theorem 2.2, we will show that $\mathscr{G}(t)$ is analytic in $\sqrt{t}$ by the use of the Analytic Implicit Function

Theorem. To this end we consider the Banach space

$$\mathscr{B} = C^{2,\alpha}(M \sim U) \cap \{u \colon y^\alpha |u|_{C^0},\ y^\alpha |u|_{C^\alpha},\ y^\alpha |y^2 \Delta_{\sigma_0} u|_{C^0},\ y^\alpha |y^2 \Delta_{\sigma_0} u|_{C^\alpha} \text{ bounded on } U\}$$

with norm

$$|u|_{\mathscr{B}} = |u|_{C^{2,\alpha}(M\sim U)} + \sup_{z\in U} y^\alpha |u|_{C^0} + \sup_{z\in U} y^\alpha |u|_{C^\alpha} + y^\alpha |y^2 \Delta_{\sigma_0} u|_{C^0(U)} + \sup_{z\in U} y^\alpha |y^2 \Delta_{\sigma_0} u|_{C^\alpha}.$$

Here we have identified the neighborhood of the cusp with its isometric copy as a strip of the upper half-plane, and we have computed C^0 and C^α norms with respect to the Euclidean (σ_0) structure of that strip (we then explicitly display the factors of y that appear). Similarly we define

$$\mathscr{C} = C^\alpha(M \sim U) \cap \{f \colon y^\alpha |f|_{C^0},\ y^\alpha |f|_\alpha \text{ bounded on } U\}$$

with norm

$$|f|_{\mathscr{C}} = |u|_{C^\alpha(M\sim U)} + \sup_{z\in U} y^\alpha |f|_{C^0} + \sup_{z\in U} y^\alpha |f|_{C^\alpha}.$$

We then consider the map

$$F(\mathscr{G}, t) \colon \mathscr{B} \times \mathbf{C} \to \mathscr{C}$$

defined by

$$F(\mathscr{G}, t) = \frac{\Delta_\sigma \mathscr{G}}{\mathscr{K}(t)} - 2(e^{\mathscr{G}} - 1)\left[1 + \frac{t^2 a_{-2}^2 e^{-\mathscr{G}}}{\sigma^2 \mathscr{K}(t)^2}\right] - \frac{2t^2(|\Phi_0|^2 - a_{-2}^2)e^{-\mathscr{G}}}{\sigma^2 \mathscr{K}(t)} - \frac{\alpha(t)}{\mathscr{K}(t)}.$$

Since $\mathscr{K}(0) \equiv 1$, we see that $F(0, 0) = 0$; since $\mathscr{K}(t)$ is analytic in $\sqrt{t}$, we see that F is an analytic operator in $\sqrt{t}$. Before proceeding, we observe that in $P_a \cong [0, 1) \times [a, \infty)$, in view of the definition (5.1), the conformal invariant $\sigma K(t)$ satisfies $0 < C^{-1} < \sigma K(t) < C < \infty$ and $y^\alpha |\sigma K(t)|_{C^\alpha(U)} < C$ for $t > 0$; also, using the conformal map $z = (-i/2\pi)\log w$ from the punctured disk $\{0 < |w| < e^{-2\pi a}\}$ to P_a we see that $|\Phi_0|^2 - a_{-2}^2 = O(e^{-2\pi y})$. Finally we recall that $\alpha(t)$ is supported off P_a.

By considering separately the cases $t = 0$ and $t > 0$, we see that F is a bounded operator.

Next we consider the map of Banach spaces $dF_{\mathscr{G}}(\mathscr{G}(0), 0)\colon \mathscr{B} \to \mathscr{C}$ given by $dF_{\mathscr{G}}(\mathscr{G}(0), 0)[\psi] = \Delta_\sigma \psi - 2\psi$. We need to show that $dF_{\mathscr{G}}(\mathscr{G}(0), 0)[\psi]$ is invertible. So let $f \in \mathscr{C}$ and let U_r denote $U \cap M_r^c$. We first build a solution ψ to $\Delta_\sigma \psi - 2\psi = f$ by considering solutions ψ_r, to the problem $\Delta_\sigma \psi_r - 2\psi_r = f$ in $M \sim U_r$ with boundary conditions $\psi_r|_{\partial(M\sim U_r)} = 0$. By the maximum principle, $|\psi_r|_{C^0(M\sim U_r)} \le \frac{1}{2}|f|_{C^0(M\sim U_r)}$. Thus, for r_0 fixed, and for $s > r_0$, standard elliptic theory [8], Theorem 6.6 gives

$$|\psi_s|_{C^{2,g}(M\sim U_{r_0})} < C(r_0)\left[|f|_{C^\alpha(M\sim U_{r_0})} + |\psi_s|_{C^0(M\sim U_{r_0})}\right]$$
$$< C(r_0)\left[|f|_{\mathscr{C}} + \tfrac{1}{2}|f|_{C^0(M\sim U_s)}\right] < \frac{3}{2}C(r_0)|f|_{\mathscr{C}}.$$

Thus, we get convergence in $C^{2,\alpha'}$ of $\psi_s|_{(M\sim U_{r_0})}$, as $s \to \infty$, to a solution ψ of $\Delta\psi - 2\psi = f$; also $|\psi|_{C^0(M)} \le \frac{1}{2}|f|_{C^0(M)}$.

We next estimate $|\psi|_{\mathscr{B}}$, which we defined as a sum of five terms.

First we claim that

$$\sup_U y^\alpha|\psi| \le K(\alpha)\sup_U y^\alpha|f| = C \tag{5.5}$$

as $y \to \infty$. Since $y^\alpha|f| \le C$, we know that if ψ_+ satisfies $\Delta_\sigma\psi_+ - 2\psi_+ = -Cy^{-\alpha}$ on $P_a \sim P_b$ with $\psi_+|_{\partial(P_a\sim P_b)} = \frac{1}{2}|f|_{C^0(M)} > \psi$, then by the maximum principle, $\psi_+ > \psi$.

On the other hand, as $b \to \infty$, the solution to the above converges to

$$\psi_+^* = Cy^{-\alpha}/(2-\alpha^2-\alpha) + y^{-1}\left(\tfrac{1}{2}|f|_{C^0(U)} - C/(2-\alpha^2-\alpha)\right).$$

Similarly,

$$\psi > \psi_-^* = -Cy^{-\alpha}/(2-\alpha^2-\alpha) + y^{-1}(-\frac{1}{2}|f|_{C^0(U)} + C/(2-\alpha^2-g a)),$$

the limit as $b \to \infty$ of solutions ψ_- satisfying $\Delta_\sigma\psi_- - 2\psi_- = Cy^{-\alpha}$ with $\psi_-|_{\partial(P_a\sim P_b)} = -\frac{1}{2}|f|_{C^0(U)} < \psi$. So $y^\alpha|\psi| < \max(y^\alpha|\psi_+|, y^\alpha|\psi_-|) \le K(\alpha)\sup_U y^\alpha|f|$ and the claim follows.

Next we estimate $y^\alpha|\psi|_{C^\alpha(U)}$. To do this we consider the equation $\Delta_\sigma\psi = 2\psi + f$ as an equation on the flat cylinder $(U, \sigma_0|dz|^2 = |dz|^2)$; this means that we write the equation as $\Delta_{\sigma_0}\psi = y^{-2}(2\psi + f)$. So on the flat cylinder $(U_r \sim U_{r+1}, |dz|^2)$ of modulus 1 (independent of r), we have the trivial estimate [8, Problem 4.8]

$$|\psi|_{C^\alpha} \le C_1(U_1 \sim U_2, \alpha)(|\psi|_{C^0} + |y^{-2}(2\psi + f)|_{C^0})$$

so that

$$
\begin{aligned}
y^{\alpha}|\psi|_{C^{\alpha}} &\le C_1(y^{\alpha}|\psi|_{C^0} + y^{\alpha-2}|2\psi + f|_{C^0}) \\
&\le C_1(y^{\alpha}|\psi|_{C^0} + 2y^{\alpha-2}|\psi|_{C^0} + y^{\alpha-2}|f|_{C^0}) \\
&\le C_3 \sup y^{\alpha}|f|.
\end{aligned}
\tag{5.6}
$$

For the third and fourth terms of $|\psi|_{\mathscr{B}}$, we notice that

$$
\begin{aligned}
|y^2\Delta_{\sigma_0}\psi|_{C^0(M)} &= |2\psi + f|_{C^0(M)} \le 2|\psi|_{C^0(M)} + |f|_{C^0(M)} \\
&\le \left(2\left(\tfrac{1}{2}\right) + 1\right)|f|_{C^0(M)};
\end{aligned}
\tag{5.7}
$$

moreover, $y^{\alpha}|y^2\Delta_{\sigma_0}\psi|_{C^0(M)} \le 2y^{\alpha}|f|_{C^0(M)}$.

Finally,

$$
\begin{aligned}
y^{\alpha}|y^2\Delta_{\sigma_0}\psi|_{C^{\alpha}} &= y^{\alpha}|2\psi + f|_{C^{\alpha}} \le y^{\alpha}(2|\psi|_{C^{\alpha}} + |f|_{C^{\alpha}}) \\
&\le 2C_3 \sup_U y^{\alpha}|f| + \sup_U y^{\alpha}|f|_{C^{\alpha}}.
\end{aligned}
\tag{5.8}
$$

Equations (5.5)–(5.8) allow us to conclude that $|\psi|_{\mathscr{B}} \le C_5|f|_{\mathscr{C}}$ so that $\|dF(\mathscr{G}(0), 0)^{-1}\| < C_5$.

Since F and $dF(\mathscr{G}(0), 0)^{-1}$ are bounded maps, the Analytic Implicit Function Theorem (again [4, Theorem 3.3.2, p. 134]) implies that there exists a family of solutions $\mathscr{G}(\sqrt{t})$ of $F(\mathscr{G}(\sqrt{t}), \sqrt{t}) = 0$, and that $\mathscr{G}(\sqrt{t})$ is complex analytic in $\sqrt{t}$, for $|\sqrt{t}|$ sufficiently small. Since $\mathscr{G}(\sqrt{t}) \in \mathscr{B}$, $y^{\alpha}|\mathscr{G}(\sqrt{t})|$ is bounded in U, so that $\mathscr{G}(\sqrt{t}) \to 0$ as $z \to p$. A maximum principle argument shows that $\mathscr{G}(\sqrt{t})$ is then the unique solution to (5.4) that vanishes into the cusp. Thus, since $\log\mathscr{H}(t) - \log\mathscr{K}(t)$ satisfies (5.4) and $\mathscr{H}(t) = \mathscr{K}(t)e^{\mathscr{G}(t)}$, we conclude that $\mathscr{H}(t)$ is also real analytic in $\sqrt{t}$. This concludes the proof of the theorem for the case of Φ_0 having a second order pole at p.

The proof in the case where Φ_0 has at worst a first order pole at p is analogous and easier, once we set $K(t) \equiv 1$ and notice that, in this case Φ_0 itself, as a function on U, decays exponentially in y. q.e.d.

Consider a hyperbolic metric ρ_t representing a point $[\rho_t]$ in $\widehat{\mathscr{N}}$ with $\Phi([\rho_t]) = (t\Phi_0, \tau)$, for some choice of τ. We want to give an explicit series development for $w(\rho_t)^*\rho_t$ in $\sqrt{t}$, as in formula (2.5). Using that $(w(\rho_t)^*\rho_t)^{2,0} t\Phi_0\, dz^2$, we see from formula (2.1) that it is enough to give an explicit series development for $\mathscr{H}(t)$.

We formally expand (2.2b) as

$$\Delta_\sigma \log \mathscr{H}(\sqrt{t}) = 2\mathscr{H}(\sqrt{t}) - 2(\sqrt{t})^2|\Phi(t)|^2/(\sigma^2\mathscr{H}(\sqrt{t})) - 2$$

to obtain $\mathscr{H}(\sqrt{t}) = \sum \mathscr{H}^{(n)}(0)(\sqrt{t})^n/n!$ in terms of the apparently nonsensical

$$\begin{aligned}
\mathscr{H}^{(1)}(0) &= (\Delta - 2)^{-1}(0), \\
\mathscr{H}^{(2)}(0) &= (\Delta - 2)^{-1}[(\Delta - 2)^{-1}(0)]^2 \\
\mathscr{H}^{(3)}(0) &= (\Delta - 2)^{-1}(3\mathscr{H}^{(2)}(0) - 2(\mathscr{H}^{(1)}(0))^2), \\
H^{(4)}(0) &= (\Delta - 2)^{-1}[4\mathscr{H}^{(3)}(0)\mathscr{H}^{(1)}(0) + 3\mathscr{H}^{(2)}(0)^2 - 12\mathscr{H}^{(2)}(0)\mathscr{H}^{(1)}(0)^2 \\
&\qquad + 6\mathscr{H}^{(1)}(0)^4 - 48|\Phi_0|^2/\sigma^2];
\end{aligned}$$

here we have the problem that on a surface with k nodes, $(\Delta - 2)^{-1}$ has a $2k$-dimensional kernel, one dimension for each cusp. Let $a_{-2}^{(j)}$ denote the leading coefficient of Φ_0 at p_j; set $l_j = 2\sqrt{ta_{-2}^{(j)}}$ and $\vec{l} = (l_1, \cdots, l_n)$. To fix the appropriate value of the kernels, we observe

Corollary 5.2. *For every* $m \geq 0$,

$$\lim_{z \to p_j} \left[\frac{\partial^m}{\partial l_j^m}\bigg|_{l_j=0} \mathscr{H}(\vec{l}) - \frac{d^m}{dl^m}\bigg|_{l=0} h(l) \right] = 0,$$

where by $z \to p_j$, *we mean* $y \to \infty$ *as in the standard model* U *of the cusps around* p_j.

Proof. Suppose $p_j = p$ and compute

$$\frac{d^m}{dl^m}(\log \mathscr{H}(l) - \log h(l)) = \frac{d^m}{dl^m}\mathscr{G}(l),$$

in U. Now $\mathscr{G}(l)$ was constructed in the proof of Theorem 5.1 from the Analytic Implicit Function Theorem, whose proof is by the construction of a majorant series; thus, for $\mathscr{G}(l) = \sum \mathscr{G}^{(m)}(0)l^m/m!$ converging for $l < \varepsilon$, we have the estimate $\|\mathscr{G}^{(m)}(0)\|_{\mathscr{B}} < C\varepsilon^{-m}m!$. So for each m, we conclude that $y^\alpha|\mathscr{G}^{(m)}(0)|_{C^0(U)} < C\varepsilon^{-m}m!$ so that $\lim_{y\to\infty} |\mathscr{G}^{(m)}(0)| = 0$.

One finishes the proof by showing inductively that

$$\lim_{y \to \infty} \frac{d^m}{dl^m}(\log \mathscr{H}(l) - \log h(l)) = 0$$

is sufficient to prove $\lim_{y\to\infty}(d^m/dl^m)(\mathscr{H}(l) - h(l)) = 0$ since $h(0) \equiv \mathscr{H}(0) \equiv 1$. q.e.d.

Recall the spaces $QD_{-2,n}(\sigma)$ and $Q_n(\sigma)$ which were defined prior to Theorem 4.1. We can now prove Theorem 5.3, which was stated in the Introduction.

Proof of Theorem 5.3. Let $(M, \sigma|dz|^2; p_1, \cdots, p_k)$ be a surface with $k \leq n$ nodes which represents a point $[\sigma]$ in $\widehat{\mathscr{P}}_g$, and let $\widehat{\mathscr{N}} \subset \widehat{\mathscr{P}}_g$ be a neighborhood of $[\sigma]$ equipped with Fenchel-Nielsen coordinates $(\vec{l}, \vec{\theta}) \in \mathbf{R}^{6g-6}$. Because all of the length coordinates are finite, we can assume that the Fenchel-Nielsen coordinates were chosen with respect to a pair of pants decomposition $\{\gamma_1, \cdots, \gamma_{3g-3}\}$ that included the nodes, say, $\gamma_i = p_i$ for $1 \leq i \leq k$. A point $[\rho(\vec{l}, \vec{\theta})]$ in $\widehat{\mathscr{N}}$ represented by Fenchel-Nielsen coordinates $(\vec{l}, \vec{\theta})$ determines: (i) a harmonic map $w(\vec{l}, \vec{\theta}): (M - \bigcup\{p_i\}, \sigma|dz|^2; p_1, \cdots, p_k) \to (M, \rho(\vec{l}, \vec{\theta}))$, (ii) an element $(\Phi(\vec{l}, \vec{\theta}), \tau_1(\theta_1), \ldots, \tau_k(\theta_k)) \in Q_k(\sigma)$, and (ii) a function $\mathscr{H}(\vec{l}, \vec{\theta}) = \mathscr{H}(\Phi(\vec{l}, \vec{\theta}))$ as in the opening of this section.

We first claim that $\Phi(\vec{l}, \vec{\theta})$, regarded as a map of a neighborhood of the coordinate space $\mathbf{R}^{6g-6}$ to $QD_{-2,k}(\sigma)$, is real analytic in $\vec{l}, \vec{\theta}$. This requires some preparations. Choose a real basis $\{\Phi_1, \cdots, \Phi_k, \Phi_{k+1}, \cdots, \Phi_{6g-6-k}\}$ of $QD_{-2,k}(\sigma)$ with the property that, for $1 \leq i \leq k$, Φ_i has a second order pole at p_i with leading coefficient equal to one, and is otherwise regular or has at worst first order poles at the other nodes, and is the only basis element which has a second order pole at p_i. Thus, for $j > k$, $\int_M |\Phi_j|\, dx\, dy < \infty$, and the set $\{\Phi_{k+1}, \cdots, \Phi_{6g-6-k}\}$ forms a basis for the subspace $L^1(QD(\sigma)) \subset QD_{-2,k}(\sigma)$ of integrable holomorphic quadratic differentials on $(M, \sigma|dz|^2; p_1, \cdots, p_k)$.

We claim that we may write $\Phi(\vec{l}, \vec{\theta})$ in terms of this basis as

$$\Phi(\vec{l}, \vec{\theta}) = \sum_{i=1}^{k} (l_i^2/4)\Phi_i + \sum_{j=k+1}^{6g-6-k} t_j(\vec{l}, \vec{\theta})\Phi_j, \tag{5.9}$$

where the coefficient of Φ_i for $1 \leq i \leq k$ is the Fenchel-Nielsen coordinate corresponding to the node p_i. To see this, we first write

$$\begin{aligned} ds^2(\vec{l}, \vec{\theta}) &= w(\vec{l}, \vec{\theta})^* \rho(\vec{l}, \vec{\theta}) \\ &= \Phi(\vec{l}, \vec{\theta})\, dz^2 + \sigma(\mathscr{H}(\vec{l}, \vec{\theta}) \\ &\quad + |\Phi(\vec{l}, \vec{\theta})|^2/\mathscr{H}(\vec{l}, \vec{\theta}))\, dz\, d\bar{z} + \overline{\Phi}(\vec{l}, \vec{\theta})\, d\bar{z}^2. \end{aligned} \tag{5.10}$$

The right-hand side of (5.10) represents a hyperbolic metric cut along some simple closed geodesics γ_i (corresponding to p_i) of length l_i, and Theorem 3.11 asserts that the leading coefficient of $\Phi(\vec{l}, \vec{\theta})$ at p_i is $l_i^2/4$. This justifies the form of (5.9); when we are referring to $\Phi(\vec{l}, \vec{\theta})$ or $\mathscr{H}(\vec{l}, \vec{\theta})$ as functions of the coordinates $l_1, \cdots, l_k, t_{k=1}, \cdots, t_{6g-6-2k}$, we will write $\Phi(\vec{l}, \vec{\theta})$ as $\Phi(l_1, \cdots, l_k, \vec{t})$ and $\mathscr{H}(\vec{l}, \vec{\theta})$ as $\mathscr{H}(l_1, \cdots, l_k, \vec{t})$.

To show that $\Phi(\vec{l}, \vec{\theta})$ is analytic in $(\vec{l}, \vec{\theta})$, we first check that $(\vec{l}, \vec{\theta})$ is analytic in $(l_1, \cdots, l_k, \vec{t})$ and $\vec{\tau}$ and also that

$$\frac{\partial(l_1, \cdots, l_{3g-3}, \theta_{k+1}, \cdots, \theta_{3g-3})}{\partial(l_1, \cdots, l_k, t_{k+1}, \cdots, t_{6g-6-k})} \tag{5.11}$$

is invertible.

Consider again (5.10): by (5.9), $\Phi(\vec{l}, \vec{\theta})$ is constructed to be analytic in $(l_1, \cdots, l_k, \vec{t})$, and Theorem 5.1 shows that $\mathscr{H}(\vec{l}, \vec{\theta})$ is analytic in $(l_1, \cdots, l_k, \vec{t})$. Thus, $w(\vec{l}, \vec{\theta})^* \rho(\vec{l}, \vec{\theta})$ is analytic in $(l_1, \cdots, l_k, \vec{t})$. Since the quantities $l_{k+1}, \cdots, l_{3g-3}$, $\theta_{k+1}, \cdots, \theta_{3g-3}$ can be computed from the right-hand side of (5.10), we see that $l_{k+1}, \cdots, l_{3g-3}$ are analytic in $(l_1, \cdots, l_k, \vec{t})$. Since $(l_1, \cdots, l_k, \theta_1, \cdots, \theta_k)$ are trivially analytic in $(l_1, \cdots, l_k, \tau_1, \cdots, \tau_k)$, $(\vec{l}, \vec{\theta})$ is analytic in $(l_1, \cdots, l_k, \vec{t})$ and $\vec{\tau}$.

That the matrix (5.11) is invertible will follow once we show that

(i) $\dfrac{\partial(l_1, \cdots, l_k)}{\partial(t_{k+1}, \cdots, t_{6g-6-k})} = 0$,

(ii) $\dfrac{\partial(l_{k+1}, \cdots, l_{3g-3}, \theta_{k+1}, \cdots, \theta_{3g-3})}{\partial(t_{k+1}, \cdots, t_{6g-6-k})}$ is invertible,

(iii) $\dfrac{\partial(l_1, \cdots, l_k)}{\partial(l_1, \cdots, l_k)}$ is diagonal.

First we claim that, for $j > k$ and $i \le k$, we have $\partial l_i/\partial t_j|_\sigma = 0$. To see this, consider a length function $l_i : \widehat{\mathscr{N}} \to \mathbf{R}$, $i \le k$, and a family of metrics $ds^2(t_j)$ defined by (5.10) and (5.9) by setting $\Phi(\vec{l}, \vec{\theta}) = t_j\Phi_j \in L^1(QD(\sigma))$. Then from Theorem 4.1 and Proposition 3.13 one can compute that $ds^2(t_j)$ defines a family of noded surfaces with $l_i = 0$, $i \le k$.

Moreover, we notice that if we choose $\Phi(\vec{l}, \vec{\theta}) = \sum_{j=k+1}^{6g-6-k} t_j \Phi_j$ in (5.9) so that $\sum_{i=1}^{k} (l_i^2/4)\Phi_i = 0$, the corresponding metrics $ds^2(\vec{l}, \vec{\theta})$ represent a neighborhood in $\widehat{\mathcal{N}} \cap \mathcal{B}_g$; this follows from the proof of Theorem 4.1. We check that $\partial \mathcal{H}(\vec{l}, \vec{\theta})/\partial t_j|_\sigma = 0$ for all j so that

$$\frac{\partial(l_{k+1}, \cdots, l_{3g-3}, \theta_{k+1}, \cdots, \theta_{3g-3})}{\partial(t_{k+1}, \cdots, t_{6g-6-k})} \tag{5.12}$$

has entries depending only on the holomorphic (2.0) parts of $ds^2(\vec{l}, \vec{\theta})$. Then since, for the Teichmüller space of a punctured surface, the gradients of the Fenchel-Nielsen coordinates can be represented as a basis of $L^1(QD(\sigma))$ [23], we conclude that (5.12) is invertible.

Finally we observe that for $a, i \le k$, we have $\partial l_a/\partial l_i|_\sigma = \delta_i^a$. While this follows from the construction of (5.9) and (5.10), one could also easily check that $l_a = O(l_i)\delta_i^a$ by first using Corollary 5.2 to directly compute the Taylor series of $ds^2(\vec{l}, \vec{\theta})$, and then observing that because $K(ds^2(\vec{l}, \vec{\theta})) = -1$, the function l_a is determined by the length and geodesic curvature of a simple curve homotopic to p_i. This concludes the proof that (5.11) is invertible.

Now since $(l_1, \cdots, l_{3g-3}, \theta_{k+1}, \cdots, \theta_{3g-3})$ are analytic in $l_1, \cdots, l_k$, $t_{k+1}, \cdots, t_{6g-6-k}$ and (5.11) is invertible, $l_1, \cdots, l_k, t_{k+1}, \cdots, t_{6g-6-k}$ are analytic in $l_1, \cdots, l_{3g-3}$, $\theta_{k+1}, \cdots, \theta_{3g-3}$. Since $\Phi(\vec{l}, \vec{\theta})$ is analytic in $(l_1, \cdots, l_k, t_{k+1}, \cdots, t_{6g-6-k})$, we conclude that $\Phi(\vec{l}, \vec{\theta})$ is analytic in $(\vec{l}, \vec{\theta})$. By Theorem 5.1, $\mathcal{H}(\vec{l}, \vec{\theta})$ is analytic in $l_1, \cdots$, l_k, $t_{k+1}, \cdots, t_{6g-6-k}$, hence in $(\vec{l}, \vec{\theta})$. Finally, consider formula (5.10). Since $\Phi(\vec{l}, \vec{\theta})$ and $\mathcal{H}(\vec{l}, \vec{\theta})$ are analytic in $(\vec{l}, \vec{\theta})$, so is $w(\rho(\vec{l}, \vec{\theta}))^* \rho(\vec{l}, \vec{\theta})$, proving the theorem. q.e.d.

We end by describing the Taylor series in $(\vec{l}, \vec{\theta})$ of the real analytic family $ds^2(\vec{l}, \vec{\theta})$. If we consider the basis $\Phi_1, \cdots, \Phi_{6g-6-k} \in QD_{02,k}$ as basic data, then from (5.9) and (5.10) it follows that we need only describe the expansion for $\mathcal{H}(\vec{l}, \vec{\theta})$. This we gave prior to Corollary 5.2; we recall that Corollary 5.2 implied that we could fix the values of $(\Delta - 2)^{-1}$ appearing in the series for $\mathcal{H}(\vec{l}, \vec{\theta})$ by comparison with the function $h(l)$.

Corollary 5.4. *The family of metrics* $ds^2(\vec{l}\,, \vec{\theta})$ *admits an explicit real analytic expansion in terms of* $\Phi(\vec{l}\,, \vec{\theta})$ *and the operator* $(\Delta - 2)^{-1}$, *where the singular terms at the node* p_j *coming from* $\ker(\Delta - 2)$ *can be determined by comparison with the asymptotics of the expansion for* $h(l_j)$. *In particular, if* $(M\,, \sigma|dz|^2\,; p)$ *is a hyperbolic surface with one node* p, *and* Φ_0 *is a holomorphic quadratic differential with leading coefficient* $= 1$, *then*

$$ds^2(l) = \tfrac{1}{4}l^2\Phi_0\,dz^2 + \sigma(1 + \tfrac{1}{6}k(z)l^2 + O(l^3))\,dz\,d\bar{z} + \tfrac{1}{4}l^2\Phi_0\,d\bar{z}^2$$

is the expansion of for a family of hyperbolic metrics with core geodesic of length l; *here* $(\Delta_\sigma - 2)k(z) = 0$ *and* $k(z) - y^2$ *is bounded in* y *as* $y \to \infty$.

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